THEORIES OF LINEAR ORDER†

BY

MATATYAHU RUBIN

ABSTRACT

Let T be a complete theory of linear order; the language of T may contain a finite or a countable set of unary predicates. We prove the following results. (i) The number of nonisomorphic countable models of T is either finite or 2^{ω}. (ii) If the language of T is finite then the number of nonisomorphic countable models of T is either 1 or 2^{ω}. (iii) If $S_1(T)$ is countable then so is $S_n(T)$ for every n. (iv) In case $S_1(T)$ is countable we find a relation between the Cantor Bendixon rank of $S_1(T)$ and the Cantor Bendixon rank of $S_n(T)$. (v) We define a class of models $\mathscr{S}$, and show that $S_1(T)$ is finite iff the models of T belong to $\mathscr{S}$. We conclude that if $S_1(T)$ is finite then T is finitely axiomatizable. (vi) We prove some theorems concerning the existence and the structure of saturated models.

Introduction

In this paper we deal with complete theories whose models are of the type $\mathfrak{A} = \langle A, <^{\mathfrak{A}}, P_1^{\mathfrak{A}}, \cdots \rangle$ where $<^{\mathfrak{A}}$ linearly orders $\mathfrak{A}$, and $\{P_1, \cdots, P_n, \cdots\}$ is a finite or a countable set of unary predicates. In a well-known example Ehrefeucht shows that, for every positive $n \neq 2$ there is a theory T as mentioned above which has exactly n nonisomorphic countable models. In Section 6 we shall show that every such T has either finitely many nonisomorphic countable models or 2^{ω} nonisomorphic countable models. We thus obtain a complete answer to the question: given a cardinal α is there a theory T, as mentioned above, such that T has exactly α nonisomorphic countable models.

If the language of T is finite we shall sharpen our result and prove that either T is ω-categorical or T has 2^{ω} nonisomorphic countable models.

† Most of the results in this paper appeared in the author's Master of Science thesis which was prepared at the Hebrew University under the supervision of Professor H. Gaifman.

Received February 5, 1973

In Section 5 we characterize the complete theories of linear order T whose language contains a fixed finite set of unary predicates, and for which $S_1(T)$ is finite. We define the class $\mathscr{S}'$, as the smallest class of models which contains all the models with a single element and which is closed under the following operations.

(1) $s(\mathfrak{A},\mathfrak{B}) = \mathfrak{A}+\mathfrak{B}$

(2) $d(\mathfrak{A}_1,\cdots,\mathfrak{A}_n) = \sum_{r\in Q}\mathfrak{A}^r$, where $\mathbb{Q}$ is the ordered set of rationals and the family $\{\{r \mid \mathfrak{A}^r \cong \mathfrak{A}_i\} \mid i = 1,\cdots,n\}$ is a partition of $\mathbb{Q}$ consisting of dense subsets of $\mathbb{Q}$.

(3) $z(\mathfrak{A}) = \mathfrak{A}\cdot\mathbb{Z}$, where $\mathbb{Z}$ is the ordered set of the integers.

We shall show: (i) For T, as above, the following conditions are equivalent.

Condition I. $S_1(T)$ is finite.

Condition II. T has a model which belongs to $\mathscr{S}'$.

The following results will be then inferred. (ii) If $S_1(T)$ is finite then T is finitely axiomatizable. (iii) For every n the set $\{T \mid \| S_1(T) \| \leqq n\}$ is finite. (i) and (ii) are related to [6] and [4].

Rosenstein in [6] showed that if we define $\mathscr{M}$ to be the subclass of $\mathscr{S}'$ which is closed only under operations (1) and (2) then T is ω-categorial iff T has a model which belongs to $\mathscr{M}$. Rosenstein also showed that if T is ω-categorial then it is finitely axiomatizable; (ii) extends this result. Laüchli and Leonard in [4] define another class of models $\mathscr{N}$ such that $\mathscr{N} \supseteq \mathscr{S}'$. They replace the operation (3) by the poerations

(4) $$\omega(\mathfrak{A}) = \mathfrak{A}\cdot\omega,$$

(5) $$\omega^*(\mathfrak{A}) = \mathfrak{A}\cdot\omega^*,$$

and define $\mathscr{N}$ as the smallest class which is closed under (1), (2), (4), and (5). They prove that every sentence which is true in some linearly ordered set is also true in some model which belongs to $\mathscr{N}$. So they conclude that every complete theory which is finitely axiomatizable has a model which belongs to $\mathscr{N}$. However it is not true that the complete theory of every model in $\mathscr{N}$ is finitely axiomatizable, (take for instance $\omega+\omega^*$).

In Section 7 we show that if T is a complete theory of linear order with not more than ω unary predicates and $\| S_1(T) \| \leqq \omega$ then $\| S_n(T) \| \leqq \omega$ for every n. Indeed, the difficulty is in going from $S_1(T)$ to $S_2(T)$. Another question of the same nature is whether the statement that $F_1(T)$ is atomic implies that $F_2(T)$

is atomic. The answer to this question is negative; there is a complete theory of linear order T such that $F_1(T)$ is atomic and $F_2(T)$ is atomless.

We shall measure the size of $S_n(T)$ by its Cantor-Bendixon rank (see the definition in Section 7). We shall prove the following theorem: if T is a theory of linear order (again with $\leqq \omega$ unary predicates), $S_1(T)$ is countable and the rank of $S_1(T)$ is less than ν, then the rank of $S_2(T)$ is less than $\nu^4 \cdot 4 + 20$.

In Sections 2, 3, and 4 we shall develop the basic notions of this paper. In Section 2 we deal with the properties of sums, with convex submodels, and with testing formulas.

We shall call a model $\mathfrak{A}$ selfadditive (hereafter abbreviated SA) if whenever $\mathfrak{B} \equiv \mathfrak{A}$ then $\mathfrak{A} + \mathfrak{B} \succ \mathfrak{A}, \mathfrak{B}$. In Section 3 we shall prove that if $\| \mathfrak{A} \| > 1$, then $\mathfrak{A}$ is SA iff it has no convex definable subsets other than $\emptyset$ and $| \mathfrak{A} |$. We shall prove some other useful results concerning SA models.

In Section 4 we deal with saturated models. Roughly speaking, an ω-saturated model is the sum of its definable elements and some SA submodels. Each summand in this decomposition is the intersection of definable convex sets; we call these summands kernels. By means of Theorem 4.11 and Corollary 4.12 we find in which cardinalities a complete theory of linear order T has a γ-saturated model. In Theorem 4.14 we show that every α elementarily equivalent, γ-saturated infinite models of cardinality less than or equal to α have a common elementary extension of cardinality α which is γ-saturated.

We mention two open questions:

(i) Suppose we define the rank $R(X)$ of a topological space X to be the first ν such that $D^{\nu}(X) = D^{\nu+1}(X)$ where $D^{\nu}(X)$ is the Cantor Bendixon derivative of X of order ν. Is there still a function $f: \omega_1 \to \omega_1$ such that for $S_1(T)$ (not necessarily countable) $R(S_1(T)) < \nu$ implies $R(S_2(T)) < f(\nu)$?

The following question was first asked by Laüchli and Leonard in [4].

(ii) Suppose that ϕ is a sentence which is consistent with the axioms of linear order. Is there always a sentence ψ such that $\vdash \psi \to \phi$ and ψ is a finite axiomatization of a complete theory of linear order?

We believe the answer to both questions is positive.

Remark. After the manuscript was completed, S. Shelah proved that question (ii) has a positive answer, that is, there is always a ψ which is an axiomatization of a complete theory of linear order such that $\vDash \psi \vdash \phi$.

1. Preliminaries and notation

Ordinals will be denoted by letters ν, ξ, η, δ. Cardinals are defined to be initial ordinals, α, β, γ denote infinite cardinals, ω denotes the first infinite cardinal, $\alpha^{\underset{\smile}{\beta}} = \sum_{\gamma<\beta}\alpha^{\gamma}$. Natural numbers are denoted by i, j, k, l, m, n. The cardinality of a set A is denoted by $\|A\|$. If $\mathfrak{A}$ is a model we denote its cardinality by $\|\mathfrak{A}\|$, $\bar{a}, \bar{b}, \bar{P}, \cdots$ will always denote finite sequences, $\bar{a}\frown\bar{b}$ is the concatenation of $\bar{a}$ and $\bar{b}$. If X is a topological space and $A \subseteq X$ then $\mathrm{cl}(A, X)$ and $\mathrm{int}(A, X)$ denote the closure of A in X and the interior of A in X respectively. Our language is always a first order language with equality. We use $v, v_0, v_1, \cdots u, u_0, u_1, \cdots x, x_0, x_1, \cdots y, y_0, y_1, \cdots z, z_0, z_1, \cdots$ as individual variables.

If not otherwise stated, the nonlogical symbols in our language will always be: one binary predicate $<$, and a set $\{P_i \mid i < \nu\}$ of unary predicates where $0 \leqq \nu \leqq \omega$. We shall occasionally call such a language a typical language. We make the following conventions: if not otherwise stated, *a model* always means a model in a typical language; *a language* always means a typical language, L and L' denote typical languages. If $\mathfrak{A}$ is a model then $L(\mathfrak{A})$ and $T_{\mathfrak{A}}$ denote the language of $\mathfrak{A}$ and the complete theory of $\mathfrak{A}$ respectively. *A theory* always means a complete theory in a typical language which contains the axiom saying that $<$ linearly orders the whole universe. T and T' denote theories. The universe of a model $\mathfrak{A}$ is denoted by $|\mathfrak{A}|$. $\mathfrak{A} = \langle A, <, P_1 \cdots\rangle$, $\mathfrak{B} = \langle B, <, P_1 \cdots\rangle$, $\mathfrak{C} = \langle C, <, P_1, \cdots\rangle$, $\mathfrak{D} = \langle D, <, P_1, \cdots\rangle$ and $\mathfrak{K} = \langle K, <, P_1, \cdots\rangle$ denote models; subscripts and superscripts may be added. The definition of some $B \subseteq |\mathfrak{A}|$ automatically implies a definition of the model $\mathfrak{B}$, which is the submodel of $\mathfrak{A}$ whose universe is B. The interpretation of the predicate R in the model $\mathfrak{A}$ is denoted by $R^{\mathfrak{A}}$, the interpretation of $<$ in $\mathfrak{A}$ is denoted by $<^{\mathfrak{A}}$, but, naturally we omit the superscript $\mathfrak{A}$ when no confusion may arise. We always interpret $<$ as a linear order whose domain is the whole universe.

If $L \subseteq L(\mathfrak{A})$ then $\mathfrak{A} \restriction L$ denotes the model obtained from $\mathfrak{A}$ by restricting the interpretation to the symbols of L. If $a \in |\mathfrak{A}|$ we occasionally enrich $\mathfrak{A}$ by adding an individual constant to $L(\mathfrak{A})$ and interpreting it as a. It will always be understood that $\tilde{a}$ denotes this new constant and $\tilde{a}$ is interpreted in $\mathfrak{A}$ as a. If $a_1, \cdots, a_n \in |\mathfrak{A}|$ then $(\mathfrak{A}, \langle a_1, \cdots, a_n\rangle)$ denotes the model obtained from $\mathfrak{A}$ by adding individual constants to represent $a_1, \cdots, a_n$. $\mathfrak{B} \subseteq \mathfrak{A}, \mathfrak{B} \prec \mathfrak{A}, \mathfrak{B} \equiv \mathfrak{A}, \mathfrak{B} \cong \mathfrak{A}$ respectively denote that $\mathfrak{B}$ is a submodel of $\mathfrak{A}$, $\mathfrak{B}$ is an elementary submodel of $\mathfrak{A}$, $\mathfrak{B}$ is elementarily equivalent to $\mathfrak{A}$ and $\mathfrak{B}$ is isomorphic to $\mathfrak{A}$. $\mathfrak{B} \prec_g \mathfrak{A}$ means that g is an elementary embedding of $\mathfrak{B}$ into $\mathfrak{A}$; when there is no risk of confusion, we shall omit the g. $\mathfrak{B} \subseteq_g \mathfrak{A}$ means that g is a monomorphism of $\mathfrak{B}$ into $\mathfrak{A}$. If $\{\mathfrak{A}_\nu \mid \nu < \eta\}$ is a chain of models then $\bigcup_{\nu<\eta} \mathfrak{A}_\nu$ denotes the union of this chain. If for every $i \in \omega$, $\mathfrak{A}_i \subseteq_{g_i} \mathfrak{A}_{i+1}$ then $\bigcup_{i\in\omega}(\mathfrak{A}_i, g_i)$ denotes the limit of this sequence.

We define the quantifier depth of a formula and denote it by $d(\phi)$. If ϕ is atomic then $d(\phi) = 0$; $d(\exists x\, \phi) = d(\forall x\, \phi) = d(\phi) + 1$; $d(\phi \to \psi) = d(\phi \vee \psi) = d(\phi \wedge \psi) = \max(d(\phi), d(\psi))$; and $d(\sim \phi) = d(\phi)$. We say that $\mathfrak{A} \overset{n}{\equiv} \mathfrak{B}$, if for every sentence ϕ, if $d(\phi) \leqq n$, then $\phi \in T_{\mathfrak{A}}$ iff $\phi \in T_{\mathfrak{B}}$.

Let $\langle a_1, \cdots, a_n \rangle \in |\mathfrak{A}|^n$ and $\langle b_1, \cdots, b_n \rangle \in |\mathfrak{B}|^n$; we say that $\langle a_1, \cdots, a_n \rangle \cong \langle b_1, \cdots, b_n \rangle$ if the function $a_i \to b_i$ $i = 1, \cdots, n$ is an isomorphism of $\mathfrak{A}'$ and $\mathfrak{B}'$ where $\mathfrak{A}'$ and $\mathfrak{B}'$ are the submodels of $\mathfrak{A}$ and $\mathfrak{B}$ whose universes are $\{a_1, \cdots, a_n\}$ and $\{b_1, \cdots, b_n\}$ respectively.

Let Φ be a set of formulas, possibly with parameters from $|\mathfrak{A}|$, and for each $\phi \in \Phi$, ϕ has its free variables among $\{v_0, \cdots, v_{n-1}\}$. Let $\bar{a} \in |\mathfrak{A}|^n$; we write $\mathfrak{A} \models \Phi[\bar{a}]$ to denote that for every $\phi \in \Phi$, $(\mathfrak{A}, \bar{b}) \models \phi[\bar{a}]$, where $\bar{b}$ is the sequence of the parameters which occur in ϕ. We shall say that Φ is finitely satisfiable in $\mathfrak{A}$, or Φ is a type in $\mathfrak{A}$ if for every finite $\Phi_0 \subseteq \Phi$ there is $\bar{a} \in A^n$ such that $\mathfrak{A} \models \Phi_0[\bar{a}]$. Suppose $B \subseteq |\mathfrak{A}|$; we shall say that Φ is finitely satisfiable in B relative to $\mathfrak{A}$ if for every finite $\Phi_0 \subseteq \Phi$ there is $\bar{b} \in B^n$ such that $\mathfrak{A} \models \Phi[\bar{b}]$; we omit the reference to $\mathfrak{A}$ when there is no risk of confusion. Let $B \subseteq |\mathfrak{A}|$; we say that Φ is a complete type with n variables over B if Φ is a type in $\mathfrak{A}$ and for every $\phi(v_0, \cdots, v_{n-1})$ with parameters in B either $\phi \in \Phi$ or $\sim \phi \in \Phi$. We say that $\bar{a}$ realizes Φ in $\mathfrak{A}$ if $\mathfrak{A} \models \Phi[\bar{a}]$, Φ is realized in $\mathfrak{A}$ if there is $\bar{a} \in A^n$ such that $\mathfrak{A} \models \Phi[\bar{a}]$, Φ is realized in B relative to $\mathfrak{A}$ iff there is $\bar{b} \in B^n$ such that $\mathfrak{A} \models \Phi[\bar{b}]$.

Let $F_n(T)$ be the Lindenbaum algebra over T of formulas whose free variables are among $\{v_0, \cdots, v_{n-1}\}$; we identify the elements of $F_n(T)$ with the formulas of which they consist. $S_n(T)$ denotes the Stone space of $F_n(T)$. We identify $S_n(T)$ with the set of complete types with n variables over the empty set. We regard $S_n(T)$ as a topological space; the topology is always understood to be the Stone topology. Thus $S_n(T)$ is a compact Hausdorff totally disconnected space. We say that $S_n(T)$ is atomic if $F_n(T)$ is atomic, that is, $S_n(T)$ is atomic iff the set of isolated points is dense in $S_n(T)$. Trivially $S_n(T)$ has a countable basis of open sets.

Let $\mathfrak{A}$ be a model of T, and $\bar{a} \in A^n$; then $P(\bar{a}, \mathfrak{A})$ is the single type in $S_n(T)$ which is realized by $\bar{a}$. If $a \in A$ then $P(a, \mathfrak{A}) = P(\langle a \rangle, \mathfrak{A})$, if $B \subseteq A$ then $P(B, \mathfrak{A}) = \{P(b, \mathfrak{A}) \mid b \in B\}$. Let P and Q be types in $\mathfrak{A}$; we say that Q supports P if for every $\mathfrak{B} \succ \mathfrak{A}$ and every $\bar{b} \in B^n$ $\mathfrak{B} \models Q[\bar{b}]$ implies $\mathfrak{B} \models P[\bar{b}]$.

DEFINITION (i) $\mathfrak{A}$ is called γ-saturated if for every $B \subseteq A$ (such that $\|B\| < \gamma$) and for every type Φ in $\mathfrak{A}$ (with one variable and parameters in B) there is $a \in A$ such that $\mathfrak{A} \models \Phi[a]$.

(ii) $\mathfrak{A}$ is called γ-homogeneous if for every $B \subseteq A$ (such that $\|B\| < \gamma$) and for every $f : B \to A$, if $P(\langle b_1, \cdots, b_n\rangle, \mathfrak{A}) = P(\langle f(b_1), \cdots, f(b_n)\rangle, \mathfrak{A})$ for every n and $b_1, \cdots, b_n \in B$ then there is an automorphism $\tilde{f}$ of $\mathfrak{A}$ such that $\tilde{f} \supseteq f$.

Let $\mathfrak{A}$ be a model, Φ a set of formulas whose free variables are among $\{v_0, \cdots, v_n\}$ and $\bar{a} \in A^n$. We define $|\mathfrak{A}|_{\Phi,\bar{a}} = \{b \mid b \in A \text{ and } \mathfrak{A} \models \Phi[b, \bar{a}]\}$. $|\mathfrak{A}|_{\phi,\bar{a}}$, $|\mathfrak{A}|_{\Phi,a}$, $|\mathfrak{A}|_{\Phi}$, etc., abbreviate $|\mathfrak{A}|_{\{\phi\},\bar{a}}$, $|\mathfrak{A}|_{\Phi,\langle a\rangle}$, $|\mathfrak{A}|_{\Phi,\Lambda}$, etc., where Λ denotes the empty sequence. We shall write $\mathfrak{A}_{\Phi,\bar{a}}$, $\mathfrak{A}_{\phi,a}$, etc. to denote the submodel of $\mathfrak{A}$ whose universe is $|\mathfrak{A}|_{\Phi,\bar{a}}$, $|\mathfrak{A}|_{\phi,a}$, etc. respectively. The notation $\mathfrak{A}_{\Phi,\bar{a}}$ always implies that $|\mathfrak{A}|_{\Phi,\bar{a}} \neq \emptyset$.

$B \subseteq |\mathfrak{A}|$ is said to be definable over $\bar{a}$ in $\mathfrak{A}$ if there is ϕ such that $B = |\mathfrak{A}|_{\phi,\bar{a}}$. B is said to be definable in $\mathfrak{A}$ if $B = |\mathfrak{A}|_{\phi}$ for some ϕ. $b \in |\mathfrak{A}|$ is definable over $\bar{a}$ if $\{b\}$ is definable over $\bar{a}$, and b is definable in $\mathfrak{A}$ if $\{b\}$ is definable in $\mathfrak{A}$. Let $\phi(v_0, \bar{a})$ be a formula with parameters from $\mathfrak{A}$, and let $\psi(x_1, \cdots, x_l)$ be any formula. We define the relativization of ψ to ϕ to be the formula χ which expresses the fact that $x_1, \cdots, x_l$ satisfy ψ in the submodel $\mathfrak{A}_{\phi,\bar{a}}$; that is, for every $\bar{b} \in (|\mathfrak{A}|_{\phi,\bar{a}})^l$ $|\mathfrak{A}|_{\phi,\bar{a}} \models \psi[\bar{b}]$ iff $\mathfrak{A} \models \chi[\bar{b}]$.

We turn to linearly ordered sets. If not otherwise stated, $<$ denotes a linear order. $\leqq$, $>$, $\geqq$, $\not<$, $\not>$, have their conventional meanings. Let $\mathfrak{A}$ be a model, $a, b \in A$ and $a \leqq b$; then $(a, \mathfrak{A}, b)$, $[a, \mathfrak{A}, b]$, $(a, \mathfrak{A}, b]$, $(a, \mathfrak{A}|$, $[a, \mathfrak{A}|$, $|\mathfrak{A}, b)$, and $|\mathfrak{A}, b]$ respectively denote the sets $\{c \mid c \in A \text{ and } a < c < b\}$, $\{c \mid c \in A \text{ and } a \leqq c \leqq b\}$, $\{c \mid c \in A \text{ and } a < c \leqq b\}$, $\{c \mid c \in A \text{ and } c > a\}$, $\{c \mid c \in A \text{ and } c \geqq a\}$, $\{c \mid c \in A \text{ and } b > c\}$, and $\{c \mid c \in A \text{ and } b \geqq c\}$. If $B \subseteq |\mathfrak{A}|$ we define $\operatorname{conv}(B, \mathfrak{A}) = \{c \mid c \in A \text{ and } b_1 \leqq c \leqq b_2 \text{ for some } b_1, b_2 \in B\}$. When there is no risk of confusion we omit the $\mathfrak{A}$ from our notation, so the abbreviated notation will be (a, b), $(a, b]$, etc., $(a|$, $|b]$, etc., and $\operatorname{conv}(B)$.

Let $\mathfrak{A}$ be a model. We define a partial order of the subsets of $|\mathfrak{A}|$: if $B, C \subseteq |\mathfrak{A}|$ then we say that $B < C$ if $b < c$ for every $b \in B$ and every $c \in C$; we say that $B \leqq C$ if $b \leqq c$ for every $b \in B$ and $c \in C$. We say that $a < B$ if $\{a\} < B$, etc. We say that $B \not< C$ if it is not true that $B < C$. Let $B \subseteq |\mathfrak{A}|$; we say that B is bounded from above, if there is $a \in A$ such that $B \leqq a$; B is bounded from below, if there is $a \in A$ such that $a \leqq B$; B is bounded if B is bounded from above and from below.

Let $\langle I, <\rangle$ be a linearly ordered set. For every $i \in I$, let $\mathfrak{A}_i$ be in the language L. Assuming that $A_i \cap A_j = \emptyset$ for every $i \neq j$ we define $\mathfrak{A} = \Sigma_{i \in I} \mathfrak{A}_i$ to be the following model: $|\mathfrak{A}| = \bigcup_{i \in I} A_i$. $a <^{\mathfrak{A}} b$ if there is i such that $a <^{\mathfrak{A}_i} b$, or $a \in A_i$ $b \in A_j$ and $i < j$. For every unary predicate P of L, $P^{\mathfrak{A}} = \bigcup^{l}_{i \in I} P^{\mathfrak{A}_i}$. $\mathfrak{A}$ is called the sum of the $\mathfrak{A}_i$'s. By $\mathfrak{A} + \mathfrak{B}$ we always mean that the elements of B are greater than the elements of A.

Let $\mathfrak{C} = \langle C, <\rangle$ be a linearly ordered set, and $\mathfrak{A}$ a model in any typical language L. We define the model $\mathfrak{B} = \mathfrak{A} \cdot \mathfrak{C}$. $\mathfrak{B}$ is a model in the language L. $\mathfrak{B} = \Sigma_{c \in C} \mathfrak{A}_c$ where for every $c \in C$, $\mathfrak{A}_c$ is a copy of $\mathfrak{A}$.

We need a fixed notation for two particular ordered sets, namely, the integers and the rationals. $\langle \mathbb{Z}, <\rangle$ and $\langle \mathbb{Q}, <\rangle$ denote these ordered sets respectively, However, we shall always write $\mathbb{Z}$ to denote $\langle \mathbb{Z}, <\rangle$ and $\mathbb{Q}$ to denote $\langle \mathbb{Q}, <\rangle$. If $\mathfrak{A} = \langle A, <\rangle$ is a linearly ordered set then $\mathfrak{A}^* = \langle A, <^*\rangle$ denotes the ordered set obtained by reversing the order of $\mathfrak{A}$, that is, $a <^* b$ if $b < a$.

2. Convex sets and the properties of sums

DEFINITION. Let $\mathfrak{A}$ be a model and $B \subseteq A$. B is called a convex subset of $\mathfrak{A}$ if for every $b_1, b_2 \in B$, if $b \in [b_1, b_2]$ then $b \in B$. $\mathfrak{B} \subseteq \mathfrak{A}$ is called a convex submodel of $\mathfrak{A}$, if $|\mathfrak{B}|$ is a convex subset of $\mathfrak{A}$. We write $\mathfrak{B} \Subset \mathfrak{A}$ to denote $\mathfrak{B}$ is a convex submodel of $\mathfrak{A}$, and $\mathfrak{B} \Subset_g \mathfrak{A}$ to denote that g is a monomorphism of $\mathfrak{B}$ into $\mathfrak{A}$ with a convex range.

REMARK. We may give a general definition of a convex submodel: $\mathfrak{B} \subseteq \mathfrak{A}$ is a convex submodel of $\mathfrak{A}$ if for every n, k, $\bar{a} \in (A - B)^n$ and $\bar{b}_1, \bar{b}_2 \in B^k$, if $\bar{b}_1 \cong \bar{b}_2$, then $\bar{a}\,\hat{}\,\bar{b}_1 \cong \bar{a}\,\hat{}\,\bar{b}_2$. Many results in this section can be formulated so that they will hold for the more general notion of a convex submodel.

The nice properties of sums of models result from the fact that each summand is a convex submodel of the sum.

We list some well known facts about sums. Proofs may be found in [2, (5.1), (5.2)], however the reader will find it very easy to prove the lemma, using Ehrenfeucht's criterion.

THEOREM 2.1. *Let $\langle I, <\rangle$ be an ordered set; for every $i \in I$ let $\mathfrak{A}_i, \mathfrak{B}_i$ be models in the language L and $\bar{a}_i \in A_i^{k_i}$, $\bar{b}_i \in B_i^{k_i}$. Let $\mathfrak{A} = \Sigma_{i \in I} \mathfrak{A}_i$, $\mathfrak{B} = \Sigma_{i \in I} \mathfrak{B}_i$; then*

(i) *if for every $i \in I$, $\mathfrak{A}_i \equiv \mathfrak{B}_i$, then $\mathfrak{A} \equiv \mathfrak{B}$.*

(ii) *if for every* $i \in I$, $\mathfrak{A}_i \equiv \mathfrak{B}_i$ *and for some* $i_1, \cdots, i_m \in I$, $(\mathfrak{A}_{i_j}, \bar{a}_{i_j}) \equiv (\mathfrak{B}_{i_j}, \bar{b}_{i_j})$ *where* $j = 1, \cdots, m$, *then* $(\mathfrak{A}, \bar{a}_{i_1}^\frown \bar{a}_{i_2}^\frown \cdots ^\frown \bar{a}_{i_m}) \equiv (\mathfrak{B}, \bar{b}_{i_1}^\frown \bar{b}_{i_2}^\frown \cdots ^\frown \bar{b}_{i_m})$.

(iii) (i) *and* (ii) *are true when* $\equiv$ *is replaced by* $\overset{n}{\equiv}$.

(iv) *if for every* i, $\mathfrak{A}_i \prec \mathfrak{B}_i$ *then* $\mathfrak{A} \prec \mathfrak{B}$.

The following lemma can be inferred from [2, (4.7). (4.8)]. We prefer, however, to give a proof of our own.

THEOREM 2.2. *Let* $L(\mathfrak{A})$ *be finite, and* $\mathfrak{B} \subset \mathfrak{A}$. *For every* $m \geqq 0$ *and* $l \geqq 1$ *there is a finite set of formulas* $\Theta^{l,m}$ *such that for every* $k \geqq 0$ *and for every formula* $\phi(v_1, \cdots, v_l, x_1, \cdots, x_k)$ *with at most* m *quantifiers and for every* $\bar{a} \in (A-B)^k$ *there is* $\phi^*(v_1, \cdots, v_l) \in \Theta^{l,m}$ *such that for every* $\bar{b} \in B^l$ $\mathfrak{A} \models \phi[\bar{b}, \bar{a}]$ *iff* $\mathfrak{B} \models \phi^*[\bar{b}]$.

PROOF. We proof by induction on m that the theorem is true for m and for every l. For $m = 0$, let $\Theta^{l,0}$ be a finite set of formulas without quantifiers with free variables among $\{v_1, \cdots, v_l\}$, such that for every formula without quantifiers $\psi(v_1, \cdots, v_l)$ there is $\phi \in \Theta^{l,0}$ such that $\dashv \phi \leftrightarrow \psi$. Obviously, $\Theta^{l,0}$ has the desired properties.

Suppose the induction hypothesis is true for m. Let $\Theta = \Theta^{l,m} \cup \{\exists v_{l+1}\psi \mid \psi \in \Theta^{l+1,m}\}$ and

$$\Theta^{l,m+1} = \left\{ \bigvee_{i=1}^{n} \psi_i \,\middle|\, \{\psi_1, \cdots, \psi_n\} \subseteq \Theta \right\} \cup \left\{ \sim \bigvee_{i=1}^{n} \psi_i \,\middle|\, \{\psi_1, \cdots, \psi_n\} \subseteq \Theta \right\}.$$

Certainly $\Theta^{l,m+1}$ is finite. Let $\chi(v_1, \cdots, v_l, x_1, \cdots, x_k)$ be a formula with $m+1$ quantifiers, and let $\bar{a} \in (A-B)^k$. We may assume that $\chi \equiv \exists y \phi(v_1, \cdots, v_l, y, x_1, \cdots, x_k)$. By the induction hypothesis there is a finite set of formulas $\{\psi_1(v_1, \cdots, v_l), \cdots, \psi_r(v_1, \cdots, v_l)\} \subseteq \Theta^{l,m}$ such that for every $c \in A-B$ there is i such that for every $\bar{b} \in B^l$, $\mathfrak{A} \models \phi[\bar{b}, c, \bar{a}]$ iff $\mathfrak{B} \models \psi_i[\bar{b}]$. By the induction hypothesis there is $\psi(v_1, v_2, \cdots, v_l, y) \in \Theta^{l+1,m}$ such that for every $\bar{b} \in B^l$ and $c \in B$, $\mathfrak{A} \models \phi[\bar{b}, c, \bar{a}]$ iff $\mathfrak{B} \models \psi[\bar{b}, c]$. Let $\chi^*(v_1, \cdots, v_l) \equiv (\bigvee_{i=1}^r \psi_i) \vee \exists y \psi$, then $\chi^* \in \Theta^{l,m+1}$ and it is easily seen that for every $\bar{b} \in B^l$, $\mathfrak{A} \models \chi[\bar{b}, \bar{a}]$ iff $\mathfrak{B} \models \chi^*[\bar{b}]$. Hence the lemma is proved.

Theorem 2.2 was formulated only for finite languages. This was done because the finiteness of $\Theta^{l,m}$ was essential to the induction process. However, the main result of the theorem is also true for infinite languages.

COROLLARY 2.3. *Let* $\mathfrak{A}$ *be a model in a finite or infinite language* L; *let* $\mathfrak{B} \subset \mathfrak{A}$. *Then for every formula* $\phi(v_1, \cdots, v_l, x_1, \cdots, x_k)$ *and for every* $\bar{a} \in (A-B)^k$ *there is* $\phi^*(v_1, \cdots, v_l)$ *such that for every* $\bar{b} \in B^l$, $\mathfrak{A} \models \phi[\bar{b}, \bar{a}]$ *iff* $\mathfrak{B} \models \phi^*[\bar{b}]$.

PROOF. Since $\mathfrak{B} \upharpoonright L' \Subset \mathfrak{A} \upharpoonright L'$ for any finite sublanguage of L, L', Theorem 2.2 may be applied.

Suppose $\mathfrak{A}$, $\mathfrak{B}$, ϕ, $\bar{a}$, and ϕ^* are as in Corollary 2.3; we then call ϕ^* the testing formula of $\phi(v_1, \cdots, v_l, \bar{a})$ in the convex submodel $\mathfrak{B}$.

COROLLARY 2.4. *Let $L(\mathfrak{A})$ be finite or infinite, let B be a convex subset of $\mathfrak{A}$, $\bar{c} \in A^n$ and B definable over $\bar{c}$ in $\mathfrak{A}$. Let $\phi(v_1, \cdots, v_l, x_1, \cdots, x_k)$ be a formula and $\bar{a} \in (A-B)^k$; then there is a formula $\phi^*(v_1, \cdots, v_l, \bar{c})$ such that for every $\bar{b} \in A^l$, $\mathfrak{A} \models \phi^*[\bar{b}, \bar{c}]$ iff $\bar{b} \in B^l$ and $\mathfrak{A} \models \phi[\bar{b}, \bar{a}]$.*

PROOF. Let $\chi(v_0, \bar{c})$ define B over $\bar{c}$. Let $\phi_1(v_1, \cdots, v_l)$ be the testing formula of ϕ in $\mathfrak{B}$, let $\phi_2(v_1, \cdots, v_l, \bar{c})$ be the relativization of ϕ_1 to $\chi(v_0, \bar{c})$ and let $\phi^* \equiv \bigwedge_{i=1}^{l} \chi(\bar{c}, v_i) \wedge \phi_2$, then ϕ^* is as desired.

The next lemma roughly states that the testing formulas do not change if we replace the convex submodels $\mathfrak{B}_i$ of $\mathfrak{A}$ by elementarily equivalent models $\mathfrak{B}_i'$.

LEMMA 2.5. *For every $i \in I$, let $\mathfrak{B}_i \Subset \mathfrak{A}$, and if $i \neq j$ then $B_i \cap B_j = \emptyset$. Let $\mathfrak{B}_i' \equiv \mathfrak{B}_i$ for every $i \in I$. We assume that $(\bigcup_{i \in I} B_i') \cap A = \emptyset$. Let $\mathfrak{A}'$ be defined as follows: $A' = (A - \bigcup_{i \in I} B_i) \cup \bigcup_{i \in I} B_i'$. Let $a' \in A'$, and P be a unary predicate in $L(\mathfrak{A})$; then $a' \in P^{\mathfrak{A}'}$ iff $a' \in P^{\mathfrak{A}}$ or $a' \in P^{\mathfrak{B}_i'}$ for some $i \in I$. Let $a, b \in A'$ then: $a <^{\mathfrak{A}'} b$ iff $a <^{\mathfrak{A}} b$, or $a <^{\mathfrak{B}_i'} b$ for some $i \in I$, or $b \in B_i'$ and $a \in B_j'$ and $B_j <^{\mathfrak{A}} B_i$, or $b \in B_i'$ and $a <^{\mathfrak{A}} B_i$, or $a \in B_j'$ and $B_j <^{\mathfrak{A}} b$; then*

(i) *let $\bar{a} \in (A - \bigcup_{i \in I} B_i)^k$, $j \in I$, $\phi(v_1, \cdots, v_l, \bar{a})$ and $\phi^*(v_1, \cdots, v_l)$ be such that for every $\bar{b} \in (B_j)^l$, $\mathfrak{A} \models \phi[\bar{b}, \bar{a}]$ iff $\mathfrak{B}_j \models \phi^*[\bar{b}]$; then for every $\bar{b}' \in (B_j')^l$ $\mathfrak{A}' \models \phi[\bar{b}', \bar{a}]$ iff $\mathfrak{B}_j' \models \phi^*[\bar{b}']$.*

(ii) *if for some $j \in I$ and $\bar{a} \in (A - \bigcup_{i \in I} B_i)^n$, $B_j = |\mathfrak{A}|_{\psi, \bar{a}}$ then $B_j' = |\mathfrak{A}'|_{\psi, \bar{a}}$.*

(iii) *if $(\mathfrak{B}_j, \bar{b}) \equiv (\mathfrak{B}_j', \bar{b}')$ and $B_j = |\mathfrak{A}|_{\psi, \bar{b}}$ then $B_j' = |\mathfrak{A}'|_{\psi, \bar{b}'}$.*

(iv) *suppose again $(\mathfrak{B}_j, \bar{b}) \equiv (\mathfrak{B}_j', \bar{b}')$. Let $\bar{a} \in (A - \bigcup_{i \in I} B_i)^k$, and let $\phi(v_1, \cdots, v_l, \bar{a})$ and $\phi^*(v_1, \cdots, v_l, \bar{b})$ be such that $\mathfrak{A} \models \phi^*[\bar{c}, \bar{b}]$ iff $\bar{c} \in (B_j)^l$ and $\mathfrak{A} \models \phi[\bar{c}, \bar{a}]$; then $\mathfrak{A}' \models \phi^*[\bar{c}, \bar{b}']$ iff $\bar{c} \in (B_j')^l$ and $\mathfrak{A}' \models \phi[\bar{c}, \bar{a}]$.*

PROOF.

(i) We enrich the language of $\mathfrak{A}$ and $\mathfrak{A}'$. $\mathfrak{A}_1$ will be the model obtained by enriching $\mathfrak{A}$ and $\mathfrak{A}_1'$ is the model obtained by enriching $\mathfrak{A}'$. For every $a \in A - \bigcup_{i \in I} B_i$ let $\tilde{a}$ be an individual constant, we define $\tilde{a}^{\mathfrak{A}_1} = \tilde{a}^{\mathfrak{A}_1'} = a$. For every $i \in I$ we add a unary predicate R_i (we assume this is a new predicate) and interpret R_i as follows: $R_i^{\mathfrak{A}_1} = B_i$ and $R_i^{\mathfrak{A}_1'} = B_i'$. By Ehrenfeucht's criterion it is very easy to see that $\mathfrak{A}_1 \equiv \mathfrak{A}_1'$. Since B_j is definable in $\mathfrak{A}_1$ there is a sentence

χ in the enriched language which expresses the fact that ϕ^* is the testing formula of ϕ in $\mathfrak{B}_j$. Since $\mathfrak{A}_1 \equiv \mathfrak{A}_1'$, χ is true in $\mathfrak{A}_1'$, thus ϕ^* is the testing formula of ϕ in the submodel of $\mathfrak{A}_1'$ whose universe is $R_j^{\mathfrak{A}_1'}$, but this submodel is $\mathfrak{B}_j'$ hence (i) is proved.

(ii) and (iii) are simple corollaries of (i). The proof of (iv) is similar to the proof of (i).

Let $\mathfrak{A} \succ \mathfrak{B} \supset \mathfrak{C}$; in the rest of this section we shall find sufficient conditions for $\mathfrak{D} \subseteq \mathfrak{A}$ to be an elementary extension of $\mathfrak{B}$ or of $\mathfrak{C}$.

Definiton. Let $\mathfrak{D} \supseteq \mathfrak{B}$; we say that $\mathfrak{D}$ is a simple extension of $\mathfrak{B}$ if there are no $d_1, d_2 \in D - B$ and $b \in B$ such that $d_1 < b < d_2$.

Definition. Let $\mathfrak{A} \supseteq \mathfrak{B}$; we say that $\mathfrak{D}$ is a permissible extension of $\mathfrak{B}$ relative to $\mathfrak{A}$ if $\mathfrak{A} \supseteq \mathfrak{D} \supseteq \mathfrak{B}$ and for every $d \in D - B$

$$\{a \mid a \in A - B, \text{ and for every } b \in B \ a < b \text{ iff } d < b\} \subseteq D.$$

Theorem 2.6. *Let $\mathfrak{B} \prec \mathfrak{A}$, and let $\mathfrak{D}$ be a permissible extension of $\mathfrak{B}$ relative to $\mathfrak{A}$; then $\mathfrak{B} \prec \mathfrak{D} \prec \mathfrak{A}$.*

Proof. In order to prove that $\mathfrak{D} \succ \mathfrak{A}$ it suffices to show that if $\phi(x_1, \cdots, x_k, x)$ is any formula, $d_1, \cdots, d_k \in D$, $a \in A - D$ and $\mathfrak{A} \models \phi[d_1, \cdots, d_k, a]$, then there is $d \in D$ such that $\mathfrak{A} \models \phi[d_1, \cdots, d_k, d]$. Without loss of generality $d_1, \cdots, d_i \in D - B$ and $d_{i+1}, \cdots, d_k \in B$. We may further assume that $d_1 < \cdots < d_r < a < d_{r+1} \cdots < d_i$. Since $\mathfrak{D}$ is a permissible extension of $\mathfrak{B}$ there are b_1, $b_2 \in B$ such that $d_r < b_1 < a < b_2 < d_{r+1}$. We may assume $d_{i+1}, \cdots, d_{i+l} \in [b_1, b_2]$ and $d_{i+l+1}, \cdots$, $d_k \notin [b_1, b_2]$. Let $\phi^*(b_1, b_2, x_{i+1}, \cdots, x_{i+l}, x)$ be such that for every $a_1, \cdots a_l$, $b \in A$, $\mathfrak{A} \models \phi^*[b_1, b_2, a_1, \cdots, a_l, b]$ iff $a_1, \cdots, a_l$, $b \in [b_1, b_2]$ and $\mathfrak{A} \models \phi[d_1, \cdots, d_i$, $a_1, \cdots, a_l, d_{i+l+1}, \cdots, d_k, b]$.

$\mathfrak{A} \models \exists x \phi^*(b_1, b_2, d_{i+1}, \cdots, d_{i+l}, x)$ and all the parameters in the formula belong to B; since $\mathfrak{B} \prec \mathfrak{A}$ there is $d \in B \subseteq D$ such that $\mathfrak{A} \models \phi^*[b_1, b_2, d_{i+1}, \cdots, d_{i+l}, d]$, but then $\mathfrak{A} \models \phi[d_1, \cdots, d_k, d]$. Thus $\mathfrak{D} \prec \mathfrak{A}$, therefore also $\mathfrak{B} \prec \mathfrak{D}$.

We list some explicit cases when Theorem 2.6 may be applied.

Corollary 2.7. *Let $\mathfrak{B} \prec \mathfrak{A}$, and let $D_1 = \mathrm{conv}(B, \mathfrak{A})$.*

$D_2 = \{a \mid a \in A \text{ and } a \leqq b \text{ for some } b \in B\}$,

$D_3 = \{a \mid a \in A \text{ and } b \leqq a \text{ for some } b \in B\}$,

$D_4 = \{a \mid a \in A \text{ and } a < B\} \cup B$,

$D_5 = [b_1, b_2] \cup B \text{ where } b_1, b_2 \in B$.

Then $\mathfrak{B} \prec \mathfrak{D}_i \prec \mathfrak{A}$ for $i = 1, \cdots, 5$.

COROLLARY 2.8. *Let* $\mathfrak{A}$ *be a model of* T, $P \in S_1(T)$ *and* $\mathfrak{A}$ *omits* P; *then there is a simple extension of* $\mathfrak{A}$, $\mathfrak{B}$, *such that* $\mathfrak{B} \succ \mathfrak{A}$ *and* P *is realized in* $\mathfrak{B}$.

PROOF. Let $\mathfrak{C} \succ \mathfrak{A}$ and P realized in $\mathfrak{C}$. Suppose $c \in C$ and $\mathfrak{C} \models P[c]$. Let $\mathfrak{B}$ be the simple permissible extension of $\mathfrak{A}$ relative to $\mathfrak{C}$ such that $c \in B$; then clearly, $\mathfrak{A} \prec \mathfrak{B}$, and P is realized in $\mathfrak{B}$.

LEMMA 2.9. *Let* $\{\mathfrak{A}_\xi\}_{\xi<\nu}$ *and* $\{\mathfrak{B}_\xi\}_{\xi<\nu}$ *be such that* $\mathfrak{A}_{\xi_1} \Subset \mathfrak{A}_{\xi_2}$ *and* $\mathfrak{B}_{\xi_1} \Subset \mathfrak{B}_{\xi_2}$ *for every* $\xi_1 < \xi_2 < \nu$ *and* $\mathfrak{B}_\xi \prec \mathfrak{A}_\xi$ *for every* $\xi < \nu$. *Let* $\mathfrak{A} = \bigcup_{\xi<\nu}\mathfrak{A}_\xi$ *and* $\mathfrak{B} = \bigcup_{\xi<\nu}\mathfrak{B}_\xi$, *then* $\mathfrak{B} \prec \mathfrak{A}$.

PROOF. Let $b_1, \cdots, b_k \in B$, $a \in A$ and $\mathfrak{A} \models \phi[b_1, \cdots, b_k, a]$. It suffices to see that there is some $b \in B$ such that $\mathfrak{A} \models \phi[b_1, \cdots, b_k, b]$. Let ξ be such that $b_1, \cdots, b_k$, $a \in A_\xi$. Since $\mathfrak{A}_\xi \Subset \mathfrak{A}$ there is a testing formula ϕ^* such that for every $a_1, \cdots, a_k$, $a_{k+1} \in A_\xi$, $\mathfrak{A} \models \phi\,[a_1, \cdots, a_{k+1}]$ iff $\mathfrak{A}_\xi \models \phi^*[a_1, \cdots, a_{k+1}]$. Thus, $\mathfrak{A}_\xi \models \exists x\, \phi^*(b_1, \cdots, b_k, x)$. Since $\mathfrak{B}_\xi \prec \mathfrak{A}_\xi$ there is $b \in B_\xi$ such that $\mathfrak{A}_\xi \models \phi^*[b_1, \cdots, b_k, b]$, but then $\mathfrak{A} \models \phi[b_1, \cdots, b_k, b]$ and the lemma is proved.

THEOREM 2.10. *Let* $\mathfrak{B} \prec \mathfrak{A}$ *and* $\mathfrak{B} = \mathfrak{B}_1 + \mathfrak{C} + \mathfrak{B}_2$, *then*

(i) *if* $\mathfrak{D}$ *is a permissible extension of* $\mathfrak{C}$ *relative to* $\mathfrak{A}$ *and* $D \subseteq \mathrm{conv}(C, \mathfrak{A})$, *then* $\mathfrak{C} \prec \mathfrak{D}$.

(ii) *if* $\mathfrak{D}$ *is a permissible extension of* $\mathfrak{B}_1$ *relative to* $\mathfrak{A}$ *and* $D \subseteq \{a \mid a \in A$ *and* $a \leqq b$ *for some* $b \in B_1\}$ *then* $\mathfrak{B}_1 \prec \mathfrak{D}$.

(iii) *if* $\mathfrak{D}$ *is a permissible extension of* $\mathfrak{B}_2$ *relative to* $\mathfrak{A}$ *and* $D \subseteq \{a \mid a \in A$ *and* $b \leqq a$ *for some* $b \in B_2\}$ *then* $\mathfrak{B}_2 \prec \mathfrak{D}$.

PROOF. (i) Let $\mathfrak{C}_1$ be the submodel of $\mathfrak{A}$ whose universe is $\mathrm{conv}(C, \mathfrak{A})$. It suffices to show that $\mathfrak{C} \prec \mathfrak{C}_1$, for if this is so, and $\mathfrak{D}$ satisfies the conditions of (i) then $\mathfrak{D}$ is a permissible extension of $\mathfrak{C}$ relative to $\mathfrak{C}_1$ and by Theorem 2.6 $\mathfrak{C} \prec \mathfrak{D} \prec \mathfrak{C}_1$. Let $\{c_\xi\}_{\xi<\nu}, \{d_\xi\}_{\xi<\nu}$ be sequences in B, such that $c_{\xi_2} \leqq c_{\xi_1} \leqq d_{\xi_1} \leqq d_{\xi_2}$ for every $\xi_1 < \xi_2 < \nu$ and $\bigcup_{\xi<\nu}[c_\xi, \mathfrak{B}, d_\xi] = C$.

For every $\xi < \nu$ let $\mathfrak{C}_\xi$ and $\mathfrak{A}_\xi$ be the submodels of $\mathfrak{A}$ whose universes are $[c_\xi, \mathfrak{B}, d_\xi]$ and $[c_\xi, \mathfrak{A}, d_\xi]$ respectively. Then $\mathfrak{C}_\xi = \mathfrak{B}_{\chi,\langle c_\xi, d_\xi\rangle}$ and $\mathfrak{A}_\xi = \mathfrak{A}_{\chi,\langle c_\xi, d_\xi\rangle}$, where $\chi \equiv c_\xi \leqq v_0 \leqq d_\xi$. Since $\mathfrak{C} \prec \mathfrak{A}$ it is easy to see that $\mathfrak{C}_\xi \prec \mathfrak{A}_\xi$. $\mathfrak{C} = \bigcup_{\xi<\nu}\mathfrak{C}_\xi$ and $\mathfrak{C}_1 = \bigcup_{\xi<\nu}\mathfrak{A}_\xi$, thus by Lemma 2.9 $\mathfrak{C} \prec \mathfrak{C}_1$. The proof of (ii) and (iii) is analogous.

3. Selfadditive models

We shall deal in this section with a special class of models which will be called selfadditive SA. There are two facts that make SA models important: (i) if $\mathfrak{A}, \mathfrak{B}$

are elementarily equivalent SA models, then $\mathfrak{A}+\mathfrak{B} \succ \mathfrak{A},\mathfrak{B}$. (ii). Roughly speaking, every saturated model is the sum of SA models.

LEMMA 3.1. *Let* $\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{B} + \mathfrak{A}_2$, *and suppose* $\mathfrak{A} \succ \mathfrak{A}_1, \mathfrak{A}_2$, *then* $\mathfrak{A} \succ \mathfrak{A}_1 + \mathfrak{A}_2$.

PROOF. It suffices to show that for every formula $\phi(x_1, y, x_2)$ and for every $a_i \in A_i$ where $i = 1,2$, if there is $b \in B$ such that $\mathfrak{A} \models \phi[a_1, b, a_2]$, then there is $c \in A_1 \cup A_2$ such that $\mathfrak{A} \models \phi[a_1, c, a_2]$. Let $\phi^*(x_1, y)$ be the testing formula for $\phi(x_1, y, a_2)$ in $\mathfrak{A}_1$. Suppose by way of contradiction there is $b \in B$ such that $\mathfrak{A} \models \phi[a_1, b, a_2]$, but there is no $c \in A_1$ such that $\mathfrak{A} \models \phi[a_1, c, a_2]$; then $\mathfrak{A}_1 \models \sim \exists y\, \phi^*(a_1, y)$. Since $\mathfrak{A} \succ \mathfrak{A}_1$, $\mathfrak{A} \models \sim \exists y\, \phi^*(a_1, y)$. Thus $\mathfrak{A} \models \phi[a_1, b, a_2] \wedge \sim \exists y\, \phi^*(a_1, y)$. We prove that

$$\mathfrak{A} \models \forall z \exists x_1 \exists y (x_1 < z \wedge y < z \wedge \phi(x_1, y, a_2) \wedge \sim \exists u\, \phi^*(x_1, u)) \overset{\text{def}}{=} \chi .$$

If not, then since $\mathfrak{A}_2 \prec \mathfrak{A}$, $\mathfrak{A}_2 \models \sim \chi$. Thus there is $d \in A_2$ such that

$$\mathfrak{A}_2 \models \forall x_1 \forall y ((x_1 < d \wedge y < d) \rightarrow \sim(\phi(x_1, y, a_2) \wedge \sim \exists u\, \phi^*(x_1, u))).$$

Since $\mathfrak{A} \succ \mathfrak{A}_2$ the same holds in $\mathfrak{A}$; but this is impossible since $a_1 < d$, $b < d$ and nevertheless $\mathfrak{A} \models \phi(a_1, b, a_2) \wedge \sim \exists u\, \phi^*(a_1, u)$. Let $d \in A_1$; we showed that

$$\mathfrak{A} \models \exists x_1 \exists y (x_1 < d \wedge y < d \wedge \phi(x_1, y, a_2) \wedge \sim \exists u\, \phi^*(x_1, y));$$

since ϕ^* is the testing formula of $\phi(x_1, y, a_2)$ in $\mathfrak{A}_1$ and $d \in A_1$ this is impossible. So there must be some $c \in A_1$ such that $\mathfrak{A} \models \phi[a_1, c, a_2]$, and the lemma is proved.

THEOREM 3.2. *Suppose* T *has a model containing more than one point, then the following conditions are equivalent.*

(i) *If* $\mathfrak{A}$ *is a model of* T *then* $\mathfrak{A}$ *has no definable convex subset other than* $|\mathfrak{A}|$ *and* $\varnothing$.

(ii) *There are models of* T, $\mathfrak{A}$, *and* $\mathfrak{A}'$ *such that* $\mathfrak{A} + \mathfrak{A}' \succ \mathfrak{A}, \mathfrak{A}'$ *and* $\mathfrak{A} \cong \mathfrak{A}'$.

(iii) *For every* $\mathfrak{A}, \mathfrak{B}$, *if* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are models of* T, *then* $\mathfrak{A}+\mathfrak{B} \succ \mathfrak{A},\mathfrak{B}$.

PROOF. Clearly (iii) $\Rightarrow$ (ii); it is also easy to see that (ii) $\Rightarrow$ (i). Suppose that (i) holds; we show that (iii) holds. It is easy to see that the models of T have no first and last elements. Further, for every $\phi(x_1, \cdots, x_n)$ the sentence

$$(1) \quad \exists x_1 \cdots \exists x_n \phi(x_1, \cdots, x_n) \rightarrow \forall y \exists x_1 \cdots \exists x_n \left(\bigwedge_{i=1}^{n} (x_i > y) \wedge \phi(x_1, \cdots, x_n) \right)$$

belongs to T.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be models of T; we assume that $A \cap B = \varnothing$. Let

$$\Sigma = D(\mathfrak{A}) \cup D(\mathfrak{B}) \cup \{\tilde{a} < \tilde{b} \mid a \in A \text{ and } b \in B\},$$

where $D(\mathfrak{A})$ denotes the complete diagram of $\mathfrak{A}$. If Σ is not consistent then there is a finite $\Sigma_0 \subseteq \Sigma$ such that $D(\mathfrak{A}) \cup \Sigma_0$ is not consistent. Without loss of generality $\Sigma_0 = \{\phi(\tilde{b}_1, \cdots, \tilde{b}_n), \tilde{a} < \tilde{b}_1, \cdots, \tilde{a} < \tilde{b}_n\}$ where $a \in A$ and $b_i \in B$ where $i = 1, \cdots, n$. Since the $\tilde{b}_i$ do not occur in $D(\mathfrak{A})$,

$$D(\mathfrak{A}) \vdash \forall x_1, \cdots \forall x_n \left(\bigwedge_{i=1}^{n} (\tilde{a} < x_i) \rightarrow \sim \phi(x_1, \cdots, x_n)\right),$$

but this contradicts (1). Thus Σ is consistent.

Let $\mathfrak{C}$ be a model of Σ. We may assume that $\mathfrak{A}, \mathfrak{B} \prec \mathfrak{C}$. Let $\bar{A} = \{c \mid c \leqq a$ for some $a \in A\}$ and $\bar{B} = \{c \mid b \leqq c$ for some $b \in B\}$. We may further assume that $\mathfrak{C} = \bar{\mathfrak{A}} + \mathfrak{D} + \bar{\mathfrak{B}}$. By Theorem 2.6 $\bar{\mathfrak{A}} \succ \mathfrak{A}$ and $\bar{\mathfrak{B}} \succ \mathfrak{B}$, thus $\bar{\mathfrak{A}} + \mathfrak{D} + \bar{\mathfrak{B}} \succ \mathfrak{A} + \mathfrak{D} + \mathfrak{B}$ and therefore $\mathfrak{A} + \mathfrak{D} + \mathfrak{B} \succ \mathfrak{A}, \mathfrak{B}$. By Lemma 3.1 $\mathfrak{A} + \mathfrak{D} + \mathfrak{B} \succ \mathfrak{A} + \mathfrak{B}$ thus $\mathfrak{A} + \mathfrak{B} \succ \mathfrak{A}, \mathfrak{B}$. Hence (i) $\Rightarrow$ (iii) and the theorem is proved.

Definition. $\mathfrak{A}$ is called selfadditive if (i), (ii), or (iii) of Theorem 3.2 holds for $T_{\mathfrak{A}}$.

Corollary 3.3. *If $\mathfrak{A}$ is SA and ω-saturated and $P \in S_1(T_{\mathfrak{A}})$ then $\{a \mid P(a, \mathfrak{A}) = P\}$ is unbounded from above and unbounded from below.*

Proof. Use (1) and the ω-saturation of $\mathfrak{A}$.

Lemma 3.4. *Let $\langle I, < \rangle$ be an ordered set. For every $i, j \in I$, $\mathfrak{A}_i \equiv \mathfrak{A}_j$ and $\mathfrak{A}_i$ is SA; then for every $J \subseteq I$ $\sum_{i \in J} \mathfrak{A}_i \prec \sum_{i \in I} \mathfrak{A}_i$.*

Proof. By induction on $\|I\|$. There is no difficulty in proving the theorem when I is finite. Let $\|I\| = \alpha \geqq \omega$ and suppose the theorem is true for every I' such that $\|I'\| < \alpha$. Let $J \subseteq I$, $\mathfrak{A} = \sum_{i \in I} \mathfrak{A}_i$ and $\mathfrak{B} = \sum_{i \in J} \mathfrak{A}_i$. We shall show that $\mathfrak{B} \prec \mathfrak{A}$. Let $\{i_\nu \mid \nu < \alpha\} = I$ be a one-to-one enumeration of I, $I_\nu = \{i_\xi \mid \xi < \nu\}$ and $J_\nu = I_\nu \cap J$, and let $\mathfrak{A}_\nu = \sum_{i \in I_\nu} \mathfrak{A}_i$, $\mathfrak{B}_\nu = \sum_{i \in J_\nu} \mathfrak{A}_i$ then by the induction hypothesis

$$\left.\begin{array}{l} \mathfrak{A}_0 \prec \mathfrak{A}_1 \prec \cdots \prec \mathfrak{A}_\nu \cdots \\ \mathfrak{B}_0 \prec \mathfrak{B}_1 \prec \cdots \prec \mathfrak{B}_\nu \cdots \end{array}\right] \text{and } \mathfrak{A}_0 \succ \mathfrak{B}_0,\ \mathfrak{A}_1 \succ \mathfrak{B}_1, \cdots, \mathfrak{A}_\nu \succ \mathfrak{B}_\nu \cdots$$

and since $\mathfrak{A} = \bigcup_{\nu < \alpha} \mathfrak{A}_\nu$ and $\mathfrak{B} = \bigcup_{\nu < \alpha} \mathfrak{B}_\nu$, $\mathfrak{B} \prec \mathfrak{A}$. Q.E.D.

For the next lemma, new notions and notations will be needed. We confine our discussion to a fixed but arbitrary typical finite language L. Let $L(\mathfrak{A}) = L$

and $\bar{a} \in A^k$; the set $\{\phi \mid \mathfrak{A} \models \phi[\bar{a}] \text{ and } d(\phi) \leqq n\}$ is called the n-type of $\bar{a}$ in $\mathfrak{A}$. Let us say that the formulas ϕ and ψ are equivalent, $\phi \sim \psi$, iff $\vdash \phi \leftrightarrow \psi$. Let $F_{n,k}$ be the set of all formulas ϕ such that the free variables of ϕ are among $\{v_0, \cdots, v_{k-1}\}$ and $d(\phi) \leqq n$, then $\sim$ partitions $F_{n,k}$ to a finite number of equivalence classes. Since $F_{n,k}$ is closed under $\wedge$, $\vee$ and $\sim$, $F_{n,k}$ may be considered as a finite Boolean algebra. Let $t_{n,k}$ be the number of atoms in $F_{n,k}$. For every n-type P there is an atom of $F_{n,k}$, ψ such that for every $\mathfrak{A}$ and every $\bar{a} \in A^k$, $\mathfrak{A} \models P[\bar{a}]$ iff $\mathfrak{A} \models \psi[\bar{a}]$.

We extend our old notation to n-types. If $\mathfrak{A}$ is a model, $a \in A$, $\bar{a} \in A^k$, $B \subseteq A$, then $P_n(\bar{a}, \mathfrak{A})$, $P_n(a, \mathfrak{A})$, $P_n(B, \mathfrak{A})$ respectively denote the n-type of $\bar{a}$, the n-type of a, and $\{P_n(b, \mathfrak{A}) \mid b \in B\}$.

Let $\bar{P} = \langle P_1, \cdots, P_k \rangle$ be a sequence of types. We shall say that $\langle a_1, \cdots, a_k \rangle$ realizes $\bar{P}$ relative to $\mathfrak{A}$ if $a_1 < a_2 < \cdots < a_k$ and for every i, $\mathfrak{A} \models P_i[a_i]$. Let $B \subseteq |\mathfrak{A}|$; we shall say that $\bar{P}$ is realized in B relative to $\mathfrak{A}$ if there is $\bar{b} \in B^k$ such that $\bar{b}$ realizes $\bar{P}$ relative to $\mathfrak{A}$.

For every n let $t_n = t_{n,2}$, We define two sequences s_n and u_n: $s_0 = u_0 = 0$,

$$s_{n+1} = 2 \cdot s_n \cdot (t_{u_n})^{s_n} + 1$$

$$u_{n+1} = s_{n+1} + u_n .$$

The next lemma is formulated so that it will be easy to prove by induction; we shall derive from it several corollaries which will be applicable in diverse cases. It is due to these facts that the formulation of Lemma 3.5 is complicated.

Lemma 3.5. *Let* $\mathfrak{A} = \mathfrak{C} + \mathfrak{B}$, $\mathfrak{A}' = \mathfrak{C}' + \mathfrak{B}'$, $\mathfrak{B} \equiv \mathfrak{B}'$, $\mathfrak{C} \equiv \mathfrak{C}'$. *Let* $b_1, \cdots, b_k \in B$, $b'_1, \cdots, b'_k \in B'$, $b_1 \leqq b_2 \leqq \cdots \leqq b_k$, $b'_1 \leqq b'_2 \leqq \cdots \leqq b'_k$, $n > 0$, *and let the following conditions hold*:

(i) $(n,1)$: *for every* i, $P_{u_n}(b_i, \mathfrak{A}) = P_{u_n}(b'_i, \mathfrak{A}')$.

(ii) $(n,2)$: *at least one of the following holds*:

(a) $P_{u_n}(b_1, \mathfrak{B}) = P_{u_n}(b'_1, \mathfrak{B}')$.

(b) $P_{u_{n-1}}(|\mathfrak{B}, b_1), \mathfrak{A}) = P_{u_{n-1}}(|\mathfrak{B}', b'), \mathfrak{A}')$

and for every $\bar{P} \in (P_{u_{n-1}}(|\mathfrak{B}, b_1), \mathfrak{A}))^{s_n}$ $\bar{P}$ *is realized in* $|\mathfrak{B}, b_1)$ *and in* $|\mathfrak{B}', b'_1)$ *relative to* $\mathfrak{A}$ *and* $\mathfrak{A}'$ *respectively.*

(iii) $(n,3)$ *for every* i, $1 \leqq i < k$, *one of the following holds*:

(a) $P_{u_n}(\langle b_i, b_{i+1} \rangle, \mathfrak{A}) = P_{u_n}(\langle b'_i, b'_{i+1} \rangle, \mathfrak{A}')$,

(b) $b_i \neq b_{i+1}$, $b'_i \neq b'_{i+1}$,

$P_{u_{n-1}}((b_i, b_{i+1}), \mathfrak{A}) = P_{u_{n-1}}((b_i', b_{i+1}'), \mathfrak{A}')$ and for every $\bar{P} \in (P_{u_{n-1}}((b_i, b_{i+1}), \mathfrak{A}))^{s_n}$, $\bar{P}$ is realized in (b_i, b_{i+1}) and $\bar{P}$ is realized in (b_i', b_{i+1}') relative to $\mathfrak{A}$ and $\mathfrak{A}'$ respectively.

Then $P_n(\langle b_1, \cdots, b_k\rangle, \mathfrak{B}) = P_n(\langle b_1', \cdots, b_k'\rangle, \mathfrak{B}')$.

PROOF. We prove by induction on n that the theorem is true for n and for every k. Suppose first that $n = 1$. It suffices to show that in Ehrenfeucht's game with one step for the models $(\mathfrak{B}, \langle b_1, \cdots, b_k\rangle)$ and $(\mathfrak{B}', \langle b_1', \cdots, b_k'\rangle)$ the second player has a winning strategy. We may assume that player I chooses $b_0 \in B$. Suppose $b_0 < b_1$. By either (ii, a) or (ii, b), for $n = 1$, there exists $b_0' \in B'$ such that $P_0(b_0', \mathfrak{B}') = P_0(b_0, \mathfrak{B})$ thus $\langle b_0', b_1', \cdots, b_k'\rangle \cong \langle b_0, b_1, \cdots, b_k\rangle$ as desired. A similar argument shows that player II wins also in the cases when $b_0 \in (b_i, b_{i+1})$ or $b_0 \in (b_k|$ or $b_0 = b_i$.

Suppose the theorem is true for n and for every k. Let $b_1 \leqq b_2 \leqq \cdots \leqq b_k$ and $b_1' \leqq b_2' \leqq \cdots \leqq b_k'$ satisfy $(n+1,1)$, $(n+1,2)$ and $(n+1,3)$. By Ehrenfeucht's criterion it suffices to show that in Ehrenfeucht's game with $n+1$ steps for the models $(\mathfrak{B}, \langle b_1, \cdots, b_k\rangle)$ and $(\mathfrak{B}', \langle b_1', \cdots, b_k'\rangle)$ player II has a winning strategy. By the induction hypothesis, it suffices to show that after the first step of the game player II can obtain two $k+1$ tuples $\langle b_0, b_1, \cdots, b_k\rangle$ and $\langle b_0', b_1', \cdots, b_k'\rangle$ such that $b_i \to b_i'$, $i = 0, 1, \cdots, k$, is an order isomorphism, and that after the two $k+1$ tuples are arranged in an increasing order they satisfy $(n,1)$, $(n,2)$ and $(n,3)$.

Without loss of generality, we may assume that player I chooses $b_0 \in B$. Suppose $b_0 < b_1$ and $(n+1, 2)(a)$ holds. Let $P = P_{u_n}(\langle b_0, b_1\rangle, \mathfrak{B})$. Let $\psi(v_0, v_1)$ be an atom of $F_{n,2}$ which generates P. Since $u_{n+1} \geqq u_n + 1$,

$$\exists v_0 \psi(v_0, v_1) \in P_{u_{n+1}}(b_1, \mathfrak{B}) = P_{u_{n+1}}(b_1', \mathfrak{B}').$$

Let b_0' be such that $\mathfrak{B}' \models \psi[b_0', b_1']$; hence b_0' is as desired. Suppose $b_0 < b_1$ but now $(n+1, 2)$(b) holds. We distinguish between three cases.

Case I. There is $\bar{P} \in (P_{u_{n-1}}(|\mathfrak{B}, b_1), \mathfrak{A}))^{s_n}$ which is not realized in (b_0, b_1). It is easy to see that there is $b_0' \in A'$ such that $P_{u_n}(\langle b_0', b_1'\rangle, \mathfrak{A}') = P_{u_n}(\langle b_0, b_1\rangle, \mathfrak{A})$. Let $\psi(v_0, v_1)$ have the following meaning: $v_0 < v_1$ and $\bar{P}$ is not realized in (v_0, v_1). A direct computation shows that $d(\psi) \leqq s_n + u_{n-1} = u_n$, hence $\psi \in P_{u_n}(\langle b_0, b_1\rangle, \mathfrak{A})$, hence $\psi \in P_{u_n}(\langle b_0', b_1'\rangle, \mathfrak{A}')$, hence $\bar{P}$ is not realized in (b_0', b_1') relative to $\mathfrak{A}'$; thus $b_0' \in B'$. Let $\bar{Q} \in (P_{u_{n-1}}(|\mathfrak{B}, b_1), \mathfrak{A}))^{s_n}$, and let $Q_0 = P_{u_n}(b, \mathfrak{A})$; then the length of $\bar{Q}^\frown\langle Q_0\rangle^\frown \bar{P}$ is less than or equal to s_{n+1}. By $(n+1, 2)$(b) $\bar{Q}^\frown\langle Q_0\rangle^\frown \bar{P}$

is realized in $|\mathfrak{B}, b_1)$ and in $|\mathfrak{B}', b_1')$, but $\langle Q_0\rangle^\frown \bar{P}$ is neither realized in $[b_0, b_1)$ nor in $[b_0', b_1')$, thus $\bar{Q}$ is realized both in $|\mathfrak{B}, b_0)$ and $|\mathfrak{B}', b_0')$ and thus the $k+1$ tuples $\langle b_0, \cdots, b_k\rangle$ and $\langle b_0', \cdots, b_k'\rangle$ satisfy $(n,1)$, $(n,2)$ and $(n,3)$.

Case II. Suppose there exists $\bar{P} \in (P_{u_{n-1}}(|\mathfrak{B}, b_1), \mathfrak{A}))^{s_n}$ which is not realized in $|\mathfrak{B}, b_0)$. Since $\mathfrak{B}' \equiv \mathfrak{B}$ there is $b_0' \in B'$ such that $P_{u_n}(b_0', \mathfrak{B}') = P_{u_n}(b_0, \mathfrak{B})$. Suppose that $\bar{P}$ is realized in $|\mathfrak{B}', b_0')$. Let $\bar{P} = \langle P_1, \cdots, P_{s_n}\rangle$, $C' < x_1' < \cdots < x_{s_n}' < b_0'$ and for every i, $\mathfrak{A}' \models P_i[x_i']$. Since $P_{u_n}(b_0, \mathfrak{B}) = P_{u_n}(b_0', \mathfrak{B}')$ and since the sequence u_n increases rapidly enough, there are $x_1, \cdots, x_{s_n}$ such that $C < x_1 < x_2 \cdots < x_{s_n} < b_0$ and such that for every i, $P_{u_{n-1}}(x_i, \mathfrak{B}) = P_{u_{n-1}}(x_i', \mathfrak{B}')$. By Theorem 2.1(iii) for every i, $(\mathfrak{A}, x_i) \overset{u_{n-1}}{\equiv} (\mathfrak{A}', x_i')$. However this contradicts the fact that $\bar{P}$ is not realized in $|\mathfrak{B}, b_0)$, hence $\bar{P}$ is not realized in $|\mathfrak{B}', b_0')$. By an argument similar to that used in Case I, we conclude that if $\bar{Q} \in (P_{u_{n-1}}(|\mathfrak{B}, b_0), \mathfrak{A}))^{s_n}$ then $\bar{Q}$ is realized in both (b_0, b_1) and (b_0', b_1'). It is easy to see that $\langle b_0, \cdots, b_k\rangle$ and $\langle b_0', \cdots, b_k'\rangle$ satisfy $(n,1)$, $(n,2)$ and $(n,3)$ as desired.

Case III. Suppose that every $\bar{P} \in (P_{u_{n-1}}(|\mathfrak{B}, b_1), \mathfrak{A}))^{s_n}$ is realized in both $|\mathfrak{B}, b_0)$ and (b_0, b_1). Let $P_0 = P_{u_n}(b_0, \mathfrak{A})$; let $\bar{Q}$ be the concatenation of all the sequences in $(P_{u_{n-1}}(|\mathfrak{B}, b_1), \mathfrak{A}))^{s_n}$: then the length of $\bar{Q}^\frown\langle P_0\rangle^\frown\bar{Q}$ is at most $2 \cdot s_n(t_{u_n})^{s_n} + 1 = s_{n+1}$. Let $\bar{x}^\frown\langle b_0'\rangle^\frown\bar{y}$ realize $\bar{Q}^\frown\langle P_0\rangle^\frown\bar{Q}$ in $|\mathfrak{B}', b_1')$ where b_0' realizes the P_0 in the middle. The sequences $\langle b_0, \cdots, b_k\rangle$ and $\langle b_0', \cdots, b_k'\rangle$ satisfy $(n,1)$ $(n,2)$ and $(n,3)$ as desired.

The cases when the first player chooses some $b \in (b_i, b_{i+1})$ are analogous to the cases already considered. The case where $b > b_k$ or $b = b_i$ for some i is trivial, hence the lemma is proved.

The following corollaries hold for models in finite or in infinite languages.

COROLLARY 3.6. *If* $\mathfrak{A} = \mathfrak{C} + \mathfrak{B}$, $b_1, b_2 \in B$, $P(b_1, \mathfrak{A}) = P(b_2, \mathfrak{A})$ *and for every* n, $P_n(|\mathfrak{B}, b_1), \mathfrak{A}) = P_n(|\mathfrak{B}, b_2), \mathfrak{A})$ *and for every* k, n *and* $\bar{P} \in (P_n(|\mathfrak{B}, b_1), \mathfrak{A}))^k$, $\bar{P}$ *is realized both in* $|\mathfrak{B}, b_1)$ *and* $|\mathfrak{B}, b_2)$ *relative to* $\mathfrak{A}$, *then* $P(b_1, \mathfrak{B}) = P(b_2, \mathfrak{B})$.

PROOF. This is almost a special case of Lemma 3.5.

COROLLARY 3.7. *If* $\mathfrak{A} = \mathfrak{C} + \mathfrak{B}$, $b_1, b_2 \in B$, $P(b_1, \mathfrak{A}) = P(b_2, \mathfrak{A})$, $P(|\mathfrak{B}, b_1), \mathfrak{A}) = P(|\mathfrak{B}, b_2), \mathfrak{A})$ *and for every* k *and* $\bar{P} \in (P(|\mathfrak{B}, b_1), \mathfrak{A}))^k$, $\bar{P}$ *is realized both in* $|\mathfrak{B}, b_1)$ *and* $|\mathfrak{B}, b_2)$, *then* $P(b_1, \mathfrak{B}) = P(b_2, \mathfrak{B})$.

PROOF. By Corollary 3.6.

COROLLARY 3.8. *If* $\mathfrak{B} \subset \mathfrak{A}$ *and* $\mathfrak{B}$ *is* SA, $\bar{b}_1, \bar{b}_2 \in B^k$ *and* $P(\bar{b}_1, \mathfrak{A}) = P(\bar{b}_2, \mathfrak{A})$, *then* $P(\bar{b}_1, \mathfrak{B}) = P(\bar{b}_2, \mathfrak{B})$.

PROOF. Let $\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{B} + \mathfrak{A}_2$. Then $\bar{b}_1, \bar{b}_2$ and the decomposition $\mathfrak{A} = \mathfrak{A}_1 + (\mathfrak{B} + \mathfrak{A}_2)$ fulfill the conditions of Lemma 3.5; hence $P(\bar{b}_1, \mathfrak{B} + \mathfrak{A}_2) = P(\bar{b}_2, \mathfrak{B} + \mathfrak{A}_2)$. Applying again Lemma 3.5, now to the decomposition of $\mathfrak{B} + \mathfrak{A}_2$ into $\mathfrak{B}$ and $\mathfrak{A}_2$, we conclude that $P(\bar{b}_1, \mathfrak{B}) = P(\bar{b}_2, \mathfrak{B})$.

COROLLARY 3.9. *If $a_1 < b_1$ and $a_2 < b_2$, $P(a_1, \mathfrak{A}) = P(a_2, \mathfrak{A})$, $P(b_1, \mathfrak{A}) = P(b_2, \mathfrak{A})$, $P((a_1, b_1), \mathfrak{A}) = P((a_2, b_2), \mathfrak{A})$ and for every n and $\bar{P} \in (P((a_1, b_1), \mathfrak{A}))^n$ $\bar{P}$ is realized both in (a_1, b_1) and (a_2, b_2), then $P(\langle a_1, b_1 \rangle, \mathfrak{A}) = P(\langle a_2, b_2 \rangle, \mathfrak{A})$.*

PROOF. This is a simpler version of Lemma 3.5. However it can be inferred from Lemma 3.5 as follows. Let $\mathfrak{C}$ contain a single element, let $\mathfrak{A}' = \mathfrak{C} + \mathfrak{A}$. Apply Lemma 3.5 to the decomposition of $\mathfrak{A}'$ into $\mathfrak{C}$ and $\mathfrak{A}$, and conclude that $P(\langle a_1, b_1 \rangle, \mathfrak{A}) = P(\langle a_2, b_2 \rangle, \mathfrak{A})$.

Corollary 3.9 has the shortcoming that it does not give any information about $P(\langle a_i, b_i \rangle, \mathfrak{A})$ when there is $\bar{P} \in (P((a_1, b_1), \mathfrak{A}))^n$ which is not realized in (a_1, b_1). Theorem 3.10 is a strengthening of Corollary 3.9 which overcomes this shortcoming.

THEOREM 3.10. *There are numbers s'_n, u'_n such that if $a_i, b_i \in A_i$, $a_i < b_i$, $i = 1, 2$, and $P_{u'n}(a_1, \mathfrak{A}_1) = P_{u'n}(a_2, \mathfrak{A}_2)$, $P_{u'n}(b_1, \mathfrak{A}_1) = P_{u'n}(b_2, \mathfrak{A}_2)$ and for every $\bar{P} \in (P_{u'n}((a_1, b_1), \mathfrak{A}_1))^{s'n}$, $\bar{P}$ is realized in (a_1, b_1) iff $\bar{P}$ is realized in (a_2, b_2), then $P_n(\langle a_1, b_1 \rangle, \mathfrak{A}_1) = P_n(\langle a_2, b_2 \rangle, \mathfrak{A}_2)$.*

We shall not give here the proof of the theorem, since it has no application in this paper. Theorem 3.10 can be further strengthened by replacing the elements a_1, b_1, a_2, b_2 by cuts. Here a cut is meant to be a subset L of $|\mathfrak{A}|$ such that if $a \in L$ then $|a] \subseteq L$. The type of L is the set of all sentences in a language containing an additional unary predicate P which are true in $(\mathfrak{A}, L)$, where L is the interpretation of P in $(\mathfrak{A}, L)$.

4. Kernels and saturated models

It will be useful to mention at this point some known facts about the order topology. We shall say that an ordered set $\langle X, < \rangle$ is complete if whenever $L \cup R = X$ and $L < R$ then either L has a maximum or R has a minimum.

THEOREM 4.1. *Let $\langle X, < \rangle$ be an ordered set; then the following conditions are equivalent:*

(i) *X (with its order topology) is compact.*

(ii) *$\langle X, < \rangle$ is complete.*

(iii) *Every subset of X has a supremum and an infimum.*

Let $\langle X, < \rangle$ be an ordered set; for every $i \in \omega$ let L_i, R_i be such that $L_i \cup R_i = X$ and $L_i < R_i$. Then $\langle L_i, R_i \rangle_{i \in \omega}$ is called a separating sequence for X, if for every $x < y$ there is i such that $x \in L_i$ and $y \in R_i$. It is easy to see that X has a countable basis of open sets iff X has a separating sequence.

The following theorem is well known.

THEOREM 4.2. *If $L(T)$ contains no unary predicates and the models of T are densely ordered, if $\gamma > \omega$ and $\mathfrak{A}$ is a γ-saturated model of T with cardinality α, then $\alpha \geqq \gamma^{\underset{\smile}{\gamma}}$.*

LEMMA 4.3. *If $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$ are γ-saturated then so is $\Sigma^n_{i=1} \mathfrak{A}_i$.*

PROOF. Let $\mathfrak{A} = \Sigma^n_{i=1} \mathfrak{A}_i$. Let P be a type over a set of cardinality less than γ. Since P is finitely satisfiable in $\mathfrak{A}$, there is i_0 such that P is finitely satisfiable in A_{i_0}. Let P^* consist of the testing formulas of the formulas of P in the convex submodel $\mathfrak{A}_{i_0}$. Since P^* is finitely satisfiable in $\mathfrak{A}_{i_0}$ and $\mathfrak{A}_{i_0}$ is γ-saturated, there is $a \in A_{i_0}$ such that $\mathfrak{A}_{i_0} \models P^*[a]$; but then $\mathfrak{A} \models P[a]$. Hence $\mathfrak{A}$ is γ-saturated.

THEOREM 4.4. *If γ is singular and $\mathfrak{A}$ is γ-saturated then $\mathfrak{A}$ is γ^+-saturated.*

PROOF. It suffices to show that every type over a set of cardinality γ is supported by a type over a set of cardinality less than γ. Let P be a type over a set $B \subseteq A$, $\| B \| = \gamma$. Without loss of generality, P is a complete type over B. If $P \ni v_0 = \tilde{b}$ for some $b \in B$ there is nothing to prove. Suppose this is not the case; let $L = \{\tilde{b} \mid \tilde{b} < v_0 \in P\}$ and $R = \{b \mid v_0 < \tilde{b} \in P\}$. Let $L' \subseteq L$ be such that for every $b \in L$ there is $b' \geqq b$ in L' and $\| L' \| < \gamma$; let $R' \subseteq R$ be such that for every $b \in R$ there is $b' \leqq b$ in R' and $\| R' \| < \gamma$. This choice is possible since $\| R \|$, $\| L \| \leqq \gamma$ and γ is singular. For every $\phi \in P$ we define ϕ^* as follows: let $b_1 \in L'$ and $b_1 \geqq b$ for every $b \in L$ which is a parameter of ϕ; let $b_2 \in R'$ and $b_2 \leqq b$ for every $b \in R$ which is parameter of ϕ; let $\phi^*(b_1, b_2, v_0)$ be the testing formula of ϕ in the convex subset $[b_1, b_2]$, that is, $\mathfrak{A} \models \phi^*[b_1, b_2, b]$ iff $b \in [b_1, b_2]$ and $\mathfrak{A} \models \phi[b]$. Note that since P is complete b_1 and b_2 are the only parameters of ϕ^*. Let $P^* = \{\phi^* \mid \phi \in P\}$. If $\mathfrak{B} \models P^*[a]$ for some $\mathfrak{B} \succ \mathfrak{A}$ then $\mathfrak{B} \models P[a]$. Thus P^* supports P, and the theorem is proved.

THEOREM 4.5. *Let $\gamma \geqq \omega$ and $\langle I, < \rangle$ be a linearly ordered set with the following property: if $I_1, I_2 \subseteq I$, $I_1 \leqq I_2$, and $\| I_1 \|, \| I_2 \| < \gamma$ then there is an $i \in I$ such that $I_1 \leqq i \leqq I_2$. For every $i \in I$ let $\mathfrak{A}_i$ be* SA *and γ-saturated, and $\mathfrak{A}_i \equiv \mathfrak{A}_j$ for every i and j; then $\Sigma_{i \in I} \mathfrak{A}_i$ is γ-saturated.*

PROOF. Let $\mathfrak{A} = \Sigma_{i \in I} \mathfrak{A}_i$. Let P be a complete type over $B \subseteq A$ and

$\| B \| < \gamma$. Let $L = \{b \mid \tilde{b} \leqq v_0 \in P\}$ and $R = \{b \mid v_0 \leqq \tilde{b} \in P\}$. We define subsets of I, $I_1 = \{i \mid A_i \cap L \neq \emptyset\}$ and $I_2 = \{i \mid A_i \cap R \neq \emptyset\}$ and let $J = \{i \mid I_1 \leqq i \leqq I_2\}$. Clearly, $J \neq \emptyset$. Let $\mathfrak{C} = \sum_{i \in J} \mathfrak{A}_i$. Since P is finitely satisfiable in C, we may assume as in Theorem 4.4 that the set of parameters of P, B is a subset of C. So since $\mathfrak{C} \prec \mathfrak{A}$ and $B \subseteq C$ it suffices to show that $\mathfrak{C} \models P[c]$ for some $c \in C$. Without loss of generality, J has a first and a last element, thus $\mathfrak{C} = \mathfrak{A}_l + \mathfrak{C}_1 + \mathfrak{A}_k$ where $l, k \in J$. Since P is finitely satisfiable in $\mathfrak{C}$, either P is finitely satisfiable in A_l or in C_1 or in A_k. Suppose P is finitely satisfiable in A_l; then for every $\phi \in P$ let ϕ^* be the testing formula of ϕ in the submodel $\mathfrak{A}_l$, let $P^* = \{\phi^* \mid \phi \in P\}$. $\mathfrak{A}_l$ is γ-saturated; P^* has fewer than γ parameters and is finitely satisfiable in $\mathfrak{A}_l$ so there is an $a \in A_l$ such that $\mathfrak{A}_l \models P^*[a]$ so $\mathfrak{C} \models P[a]$. The same argument is applied if P is finitely satisfiable in $\mathfrak{A}_k$.

Suppose P is finitely satisfiable in $\mathfrak{C}_1$. For every $\phi \in P$ let ϕ^* be the testing formula for ϕ in the submodel $\mathfrak{C}_1$. Let $P^* = \{\phi^* \mid \phi \in P\}$. Since $B \cap C_1 = \emptyset$, P^* has no parameters. Let $\{i_\nu\}_{\nu < \xi}$ be a one-to-one function from ξ onto $J - \{k, l\}$. Let $J_\nu = \{i_\eta \mid \eta < \nu\}$ and $\mathfrak{D}_\nu = \sum_{i \in J_\nu} \mathfrak{A}_i$. Then it is easy to see by induction that $\mathfrak{D}_\nu$ is ω-saturated for every ν. $\bigcup_{\nu < \xi} \mathfrak{D}_\nu = \mathfrak{C}_1$ and $\{\mathfrak{D}_\nu\}_{\nu < \xi}$ is an elementary chain, so $\mathfrak{C}_1$ is ω-saturated. P^* is finitely satisfiable in $\mathfrak{C}_1$, so there is a $c \in C_1$ such that $\mathfrak{C}_1 \models P^*[c]$, and thus $\mathfrak{A} \models P[c]$. The theorem is proved.

Let T be a complete theory. Let $F \subseteq F_1(T)$ be the Boolean algebra generated by the set of all formulas $\phi(v_0)$ in $F_1(T)$ which define convex sets in every model of T. Let $\mathscr{F}_T$ be the set of ultrafilters of F. We call the elements of $\mathscr{F}_T$ convex types. Let $\mathfrak{A}$ be a model of T. We define $\mathscr{K}_{\mathfrak{A}} = \{ |\mathfrak{A}|_\Phi | \ \Phi \in \mathscr{F}_T$ and $|\mathfrak{A}|_\Phi \neq \emptyset\}$. We call the elements of $\mathscr{K}_{\mathfrak{A}}$ kernels in $\mathfrak{A}$. It is easy to see that every element of $\mathscr{K}_{\mathfrak{A}}$ is convex.

$\mathscr{K}_{\mathfrak{A}}$ is a partition of $\mathfrak{A}$ which consists of convex subsets: it is linearly ordered by the partial order defined on the subsets of an ordered set in the introduction. Hence we shall always regard $\mathscr{K}_{\mathfrak{A}}$ as a linearly ordered set. When $\mathscr{K}_{\mathfrak{A}}$ is referred to as a topological space, it will be always understood that $\mathscr{K}_{\mathfrak{A}}$ is taken with its order topology. If $\mathfrak{A}$ is an ω-saturated model of T then the order on $\mathscr{K}_{\mathfrak{A}}$ induces an order on $\mathscr{F}_T$, that is, $\Phi < \Psi$ iff $|\mathfrak{A}|_\Phi < |\mathfrak{A}|_\Psi$. Clearly this order is independent of the choice of $\mathfrak{A}$. So, again we regard $\mathscr{F}_T$ as an ordered set. When $\mathscr{F}_T$ is considered as a topological space, the topology is understood to be the order topology.

Let $\mathfrak{A}$ be a model of T, and $a \in A$. Let $T' = T_{(\mathfrak{A}, a)}$; then the elements of $\mathscr{F}_{T'}$ will be called convex types over a. We denote $\mathscr{K}_{(\mathfrak{A}, a)}$ by $\mathscr{K}^a_{\mathfrak{A}}$, and call the elements of $\mathscr{K}^a_{\mathfrak{A}}$ kernels over a. When there is no risk of confusion, we abbreviate and write $\mathscr{K}^a$ instead of $\mathscr{K}^a_{\mathfrak{A}}$. It is inessential to our discussion whether we regard $(\mathfrak{A}, a)$ as a model with an additional individual constant $\tilde{a}$, or whether we regard it as a model with an additional unary predicate P, such that $P^{\mathfrak{A}} = \{a\}$. We may assume that always $\mathscr{K}^a_{\mathfrak{A}} = \mathscr{K}_{\mathfrak{A}'}$ where $L(\mathfrak{A}')$ consists of one binary predicate and unary predicates only. Thus whatever is proved for kernels in every typical language applies also to kernels over a.

Let $\mathfrak{A}$ be a model, $a \in A$, and $K \in \mathscr{K}_{\mathfrak{A}}$ $(K \in \mathscr{K}^a_{\mathfrak{A}})$. We say that K is definable (over a) from below if $\{b \mid b \in A \text{ and}, b > K \text{ or } b \in K\}$ is definable (over a). Definability from above is defined similarly. We list some elementary properties.

LEMMA 4.6. *Let $\mathfrak{A}$ be ω-saturated, and $K \in \mathscr{K}_{\mathfrak{A}}$. Then*

(i) *K has a successor in $\mathscr{K}_{\mathfrak{A}}$ iff K is definable from above;*
K is a successor in $\mathscr{K}_{\mathfrak{A}}$ iff K is definable from below;
K is isolated in $\mathscr{K}_{\mathfrak{A}}$ iff K is definable in $\mathfrak{A}$.

(ii) *$\mathscr{K}_{\mathfrak{A}}$ has a separating sequence.*

(iii) *$\mathscr{K}_{\mathfrak{A}}$ is complete.*

(iv) *Similar results hold for $\mathscr{K}^a_{\mathfrak{A}}$.*

The proofs are straightforward.

LEMMA 4.7. *Let $\mathfrak{A} \equiv \mathfrak{B}$; $\mathfrak{A}$ and $\mathfrak{B}$ are ω-saturated and $\Phi \subseteq F_1(T_{\mathfrak{A}})$ is finitely satisfiable. Then $\mathfrak{A}_\Phi \equiv \mathfrak{B}_\Phi$.*

PROOF. By Ehrenfeucht's criterion the proof is trivial.

LEMMA 4.8. (i) *Let $\mathfrak{A}$ be an ω-saturated model of T and $\Phi \in \mathscr{F}_T$. Then either $\mathfrak{A}_\Phi$ consists of a single element b and b is definable in $\mathfrak{A}$, or $\mathfrak{A}_\Phi$ is* SA.

(ii) *If $a \in A$ then a similar result holds for kernels over a.*

PROOF. (i) By Lemma 4.7 we may assume that $\mathfrak{A}$ is also ω-homogeneous. Let $\phi(v_1)$ be a formula; there is an $a \in |\mathfrak{A}|_\Phi$ such that $\mathfrak{A} \models \phi[a]$. Then for every $c \in |\mathfrak{A}|_\Phi$ there is a b such that $c \leqq b \in |\mathfrak{A}|_\Phi$ and $\mathfrak{A} \models \phi[b]$. If not, then (since $\mathfrak{A}$ is ω-saturated) there must be some $\psi(v_0) \in \Phi$ such that for no $b \in |\mathfrak{A}|_\psi$, $c \leqq b$ and $\mathfrak{A} \models \phi[b]$. Let $\chi(v_0) \equiv \exists x(v_0 \leqq x \wedge \psi(x) \wedge \phi(x))$; then $|\mathfrak{A}|_\chi$ is convex and $|\mathfrak{A}|_\Phi \supsetneqq |\mathfrak{A}|_{\Phi \cup \{\chi\}} \neq \emptyset$. But then Φ is not a convex type, and this is a contradiction. Hence for every $c \in |\mathfrak{A}|_\Phi$ there is b such that $c \leqq b \in |\mathfrak{A}|_\Phi$ and $\mathfrak{A} \models \phi[b]$.

Let $P \in S_1(T)$ and suppose P is realized in $|\mathfrak{A}|_\Phi$. Let $c \in |\mathfrak{A}|_\Phi$; by the preceding argument, $\{c \leqq v_0\} \cup P$ is finitely satisfiable and, since $\mathfrak{A}$ is ω-saturated, $\{c \leqq v_0\} \cup P$ is satisfiable. Thus if $P \in S_1(T)$ is realized in $|\mathfrak{A}|_\Phi$ then for every

$c \in |\mathfrak{A}|_\Phi$, P is realized in $|\mathfrak{A}|_\Phi \cap [c|$. Let $a, b \in |\mathfrak{A}|_\Phi$; let $a' \in |\mathfrak{A}|_\Phi$ be such that $a' \geqq b$ and $P(a', \mathfrak{A}) = P(a, \mathfrak{A})$. Since $\mathfrak{A}$ is ω-homogeneous there is an automorphism f of $\mathfrak{A}$ such that $f(a) = a'$. Clearly $f \restriction \mathfrak{A}_\Phi$ is an automorphism of $\mathfrak{A}_\Phi$. Thus for every $a, b \in |\mathfrak{A}|_\Phi$ there is $a' \geqq b$ and an automorphism f of $\mathfrak{A}_\Phi$ such that $f(a) = a'$. This implies that there are no convex definable subsets of $|\mathfrak{A}|_\Phi$ other than $|\mathfrak{A}|_\Phi$ and $\emptyset$. So we conclude that either $\mathfrak{A}_\Phi$ is SA or $|\mathfrak{A}|_\Phi$ consists of a single element; in the latter case, since $\mathfrak{A}$ is ω-saturated, it is obvious that this element must be definable.

To prove (ii) we have only to remark that if $\mathfrak{A}$ is ω-saturated, then so is $(\mathfrak{A}, a)$.

LEMMA 4.9. *Let $\mathfrak{A}$ be a γ-saturated model of T, and $\Phi \in \mathscr{F}_T$; then $\mathfrak{A}_\Phi$ is γ-saturated. The same is true when Φ is a convex type over some $a \in A$.*

PROOF. If $|\mathfrak{A}|_\Phi$ contains a single point then there is nothing to prove. Otherwise, let $B \subseteq |\mathfrak{A}|_\Phi$ and $\|B\| < \gamma$, let P be a complete type over B in $\mathfrak{A}_\Phi$. Let $\mathfrak{C} \succ \mathfrak{A}_\Phi$, $c_0 \in C$, and $\mathfrak{C} \models P[c_0]$. Let $\mathfrak{A}'$ be the model obtained by replacing $\mathfrak{A}_\Phi$ by $\mathfrak{C}$ in $\mathfrak{A}$.

Clearly $\mathfrak{A}' \succ \mathfrak{A}$ and $\mathfrak{A}'_\Phi = \mathfrak{C}$. Since $\mathfrak{A}$ is γ-saturated and $\|B\| < \gamma$ there is an $a \in A$ which realizes in $\mathfrak{A}$ the same complete type over B as c_0 does. Clearly $a \in |\mathfrak{A}|_\Phi$. By Lemma 4.8 $\mathfrak{A}_\Phi$ is SA, thus $\mathfrak{C}$ is SA. By Corollary 3.8, a realizes in $\mathfrak{C}$ the same complete type over B as c_0 does. Hence $\mathfrak{C} \models P[a]$ whence $\mathfrak{A}_\Phi \models P[a]$, and we conclude that $\mathfrak{A}_\Phi$ is γ-saturated.

LEMMA 4.10. *If $\mathfrak{A}_{\Phi,a}$ is SA and every $\phi \in \Phi$ defines over a a convex set and $a \notin |\mathfrak{A}|_{\Phi,a}$, then $|\mathfrak{A}|_{\Phi,a} \in \mathscr{K}^a_{\mathfrak{A}}$.*

PROOF. If $|\mathfrak{A}|_{\Phi,a} \notin \mathscr{K}^a_{\mathfrak{A}}$ then there is $\chi(a, v_0)$ which defines over a a convex set such that $|\mathfrak{A}|_{\Phi,a} \supsetneqq |\mathfrak{A}|_{\Phi\cup\{\chi\},a} \neq \emptyset$. If χ^* is the testing formula of χ in $\mathfrak{A}_{\Phi,a}$, then χ^* is without parameters, $|\mathfrak{A}_{\Phi,a}|_{\chi^*}$ is convex, and $|\mathfrak{A}|_{\Phi,a} \neq |\mathfrak{A}_{\Phi,a}|_{\chi^*} \neq \emptyset$. This contradicts the fact that $\mathfrak{A}_{\Phi,a}$ is SA. Thus $|\mathfrak{A}|_{\Phi,a} \in \mathscr{K}^a_{\mathfrak{A}}$ and the lemma is proved.

THEOREM 4.11. *Let T have an infinite model, and let $\gamma > \omega$ be regular; then T has a γ-saturated model of cardinality α iff $\alpha \geqq \gamma^{\underline{\gamma}}$.*

PROOF. Let $\mathfrak{A}$ be a γ-saturated model. We define an equivalence relation on A: $a \sim b$ iff $[a, b]$ is finite. Let A^* be the set of equivalence classes. If $a^*, b^* \in A^*$ we say that $a^* < b^*$ if, for every $a \in a^*$ and $b \in b^*$, $a < b$. Clearly $<$ linearly orders A^*. Let $\mathfrak{A}^* = \langle A^*, < \rangle$; it is easy to see that $\mathfrak{A}^*$ is densely ordered and γ-saturated. By Theorem 4.2, $\|A^*\| \geqq \gamma^{\underline{\gamma}}$, hence $\|A\| \geqq \gamma^{\underline{\gamma}}$.

Let $\alpha \geqq \gamma^{\underset{\sim}{\gamma}}$. Since γ is regular $(\gamma^{\underset{\sim}{\gamma}})^{\underset{\sim}{\gamma}} = \gamma^{\underset{\sim}{\gamma}}$. So, by [5], T has a γ-saturated model of cardinality $\gamma^{\underset{\sim}{\gamma}}$, say $\mathfrak{B}$. By Lemmas 4.8 and 4.9 there is $\mathfrak{C} \subset \mathfrak{B}$ such that $\mathfrak{C}$ is SA and γ-saturated. By Theorem 4.5, $\mathfrak{C} \cdot \langle \alpha+1, < \rangle$ is again γ-saturated. Suppose $\mathfrak{B} = \mathfrak{B}_1 + \mathfrak{C} + \mathfrak{B}_2$, and let $\mathfrak{B}_i' \equiv \mathfrak{B}_i$ where $i = 1,2$ and $\mathfrak{B}_i'$ are γ-saturated models of cardinality $\gamma^{\underset{\sim}{\gamma}}$. Let $\mathfrak{B}' = \mathfrak{B}_1' + \mathfrak{C} \cdot \langle \alpha + 1, < \rangle + \mathfrak{B}_2'$; then obviously $\mathfrak{B}' \equiv \mathfrak{B}$, $\| \mathfrak{B}' \| = \alpha$, and by Lemma 4.3, $\mathfrak{B}'$ is γ-saturated. If $\mathfrak{B} = \mathfrak{B}_1 + \mathfrak{C}$ or $\mathfrak{B} = \mathfrak{C} + \mathfrak{B}_2$ we define $\mathfrak{B}'$ similarly. If $\mathfrak{B} = \mathfrak{C}$ then $\mathfrak{C} \cdot \langle \alpha + 1, < \rangle$ is the desired γ-saturated model. Q.E.D.

COROLLARY 4.12. *If γ is singular and T has infinite models then T has a γ-saturated model of cardinality α iff $\alpha \geqq 2^{\gamma}$.*

PROOF. Combine Theorems 4.4 and 4.11.

LEMMA 4.13. *If $\mathfrak{A}$ is an ω-saturated model of T, and for every $\Phi \in \mathscr{F}_T$, $\mathfrak{A}_\Phi$ is γ-saturated, then $\mathfrak{A}$ is γ-saturated.*

PROOF. Let $B \subseteq A$, $\| B \| < \gamma$, and P a complete type over B. Let $\Phi_1 = \sup\{\Phi \mid \Phi \in \mathscr{F}_T$ and there is a $b \in |\mathfrak{A}|_\Phi$ such that $\tilde{b} \leqq v_0 \in P\}$ where the supremum is taken in $\mathscr{F}_T$. Let $\Phi_2 = \inf\{\Phi \mid \Phi \in \mathscr{F}_T$ and there is a $b \in |\mathfrak{A}|_\Phi$ such that $v_0 \leqq \tilde{b} \in P\}$. Let $C = \bigcup_{\Phi_1 < \Phi < \Phi_2} |\mathfrak{A}|_\Phi$. Then P is finitely satisfiable either in C or in $|\mathfrak{A}|_{\Phi_1}$ or $|\mathfrak{A}|_{\Phi_2}$. Suppose P is finitely satisfiable in $|\mathfrak{A}|_{\Phi_1}$. For every $\phi \in P$, let ϕ^* be the testing formula of ϕ in $\mathfrak{A}_{\Phi_1}$, and let $P^* = \{\phi^* \mid \phi \in P\}$. Then P^* is finitely satisfiable in $\mathfrak{A}_{\Phi_1}$. Since $\mathfrak{A}_{\Phi_1}$ is γ-saturated, there is an $a \in |\mathfrak{A}|_{\Phi_1}$ such that $\mathfrak{A}_{\Phi_1} \models P^*[a]$, hence $\mathfrak{A} \models P[a]$. The same argument is applied when P is finitely satisfiable in $|\mathfrak{A}|_{\Phi_2}$. Suppose P is not finitely satisfiable in $|\mathfrak{A}|_{\Phi_1} \cup |\mathfrak{A}|_{\Phi_2}$. We show that there is $\phi(v_0)$ such that $|\mathfrak{A}|_\Phi \subseteq C$ and P is finitely satisfiable in $|\mathfrak{A}|_\phi$. Let $C = \bigcup_{i \in \omega} |\mathfrak{A}|_{\phi_i}$, where for every i $|\mathfrak{A}|_{\phi_i}$ is convex. Suppose that, for no i, P is finitely satisfiable in $|\mathfrak{A}|_{\phi_i}$ and let $P_0 \subseteq P$ be finite. Then for every i, P_0 is satisfiable in $|\mathfrak{A}|_{\sim\phi_i}$. Since $\mathfrak{A}$ is ω-saturated $P_0 \cup \{\sim \phi_i \mid i \in \omega\}$ is satisfiable in $\mathfrak{A}$, that is, P_0 is satisfiable in $A - C$. Thus P is finitely satisfiable in $|\mathfrak{A}|_{\Phi_1} \cup |\mathfrak{A}|_{\Phi_2}$ which is a contradiction. So there is some i such that P is finitely satisfiable in $|\mathfrak{A}|_{\phi_i}$. For every $\phi \in P$, let $\phi^*(v_0)$ be such that $\mathfrak{A} \models \phi^*[a]$ iff $a \in |\mathfrak{A}|_{\phi_i}$ and $\mathfrak{A} \models \phi[a]$. Let $P^* = \{\phi^* \mid \phi \in P\}$. Since C contains no parameters of P, P^* is a pure type (that is, without parameters). Since $\mathfrak{A}$ is ω-saturated there is an $a \in A$ such that $\mathfrak{A} \models P^*[a]$, but then $\mathfrak{A} \models P[a]$. Thus $\mathfrak{A}$ is γ-saturated.

THEOREM 4.14. *Let $\{\mathfrak{A}^\nu \mid \nu \leqq \alpha\}$ be a set of γ-saturated infinite models of T such that for every ν, $\| \mathfrak{A}^\nu \| \leqq \alpha$. Then there is $\mathfrak{A}$ such that $\| \mathfrak{A} \| = \alpha$, $\mathfrak{A}$ is γ-saturated, and for every $\nu \leqq \alpha$, $\mathfrak{A}^\nu \prec \mathfrak{A}$.*

PROOF. For every $\Phi \in \mathscr{F}_T$ we define $\mathfrak{A}^{\Phi}$. If $|\mathfrak{A}^0|_{\Phi}$ consists of a single element then $\mathfrak{A}^{\Phi} = \mathfrak{A}^0_{\Phi}$; if not, then $\mathfrak{A}^{\Phi} = \sum_{\nu \leqq \alpha} \mathfrak{A}^{\nu}_{\Phi}$. Let $\mathfrak{A} = \sum_{\Phi \in \mathscr{F}_T} \mathfrak{A}^{\Phi}$; then it is easy to see that $\|\mathfrak{A}\| = \alpha$, that for every ν, $\mathfrak{A} \succ \mathfrak{A}^{\nu}$, for every $\Phi \in \mathscr{F}_T$, $\mathfrak{A}_{\Phi} = \mathfrak{A}^{\Phi}$, and that $\mathfrak{A}_{\Phi}$ is γ-saturated.

We shall show that $\mathfrak{A}$ is ω-saturated. Let P be a complete type over $\bar{a}$; without loss of generality, $\bar{a} = \langle a_1, a_2 \rangle$ and $\bar{a}_1 \leqq v_0 \leqq \bar{a}_2 \in P$. Suppose there is $\Phi \in \mathscr{F}_T$ such that $a_1, a_2 \in |\mathfrak{A}|_{\Phi}$. For every $\phi \in P$ let ϕ^* be the testing formula of ϕ in $\mathfrak{A}_{\Phi}$ and $P^* = \{\phi^* \mid \phi \in P\}$. P^* is finitely satisfiable in $\mathfrak{A}_{\Phi}$ and $\mathfrak{A}_{\Phi}$ is γ-saturated, thus $\mathfrak{A}_{\Phi} \vDash P^*[a]$ for some $a \in |\mathfrak{A}|_{\Phi}$, and hence $\mathfrak{A} \vDash P[a]$. Suppose a_1 and a_2 belong to distinct kernels. Let ψ be such that $\mathfrak{A} = \mathfrak{A}_{\psi} + \mathfrak{A}_{\sim\psi}$ and $a_1 \in |\mathfrak{A}|_{\psi}$, $a_2 \in |\mathfrak{A}|_{\sim\psi}$; then P is finitely satisfiable either in $|\mathfrak{A}|_{\psi}$ or in $|\mathfrak{A}|_{\sim\psi}$. Without loss of generality, P is finitely satisfiable in $|\mathfrak{A}|_{\psi}$. For every $\phi(a_1, a_2, v_0) \in P$ let $\phi^*(a_1, v_0)$ be such that $\mathfrak{A} \vDash \phi^*[a_1, a]$ iff $a \in |\mathfrak{A}|_{\psi}$ and $\mathfrak{A} \vDash \phi[a_1, a_2, a]$. Let $P^* = \{\phi^* \mid \phi \in P\}$; then P^* is a type over a_1 in $\mathfrak{A}$. Let $|\mathfrak{A}^{\nu}| \ni a_1$. Since $\mathfrak{A}^{\nu}$ is ω-saturated and $\mathfrak{A}^{\nu} \prec \mathfrak{A}$ there is an $a \in |\mathfrak{A}^{\nu}|$ such that $\mathfrak{A}^{\nu} \vDash P^*[a]$, and thus $\mathfrak{A} \vDash P[a]$. Hence $\mathfrak{A}$ is ω-saturated. By Lemma 4.13, $\mathfrak{A}$ is γ-saturated, hence $\mathfrak{A}$ is as desired. Q.E.D.

5. Theories T with finite $S_1(T)$

The discussion in this section is confined to an arbitrary but fixed finite language L. $\mathfrak{A}$ and T thus will denote a model and a theory in this language.

DEFINITION. Let $\mathfrak{A}$ be SA and $a \in A$. Let $C^a_{\mathfrak{A}}$ be the union of all the convex and bounded subsets of A which contain a and are definable over a. $\mathfrak{C}^a_{\mathfrak{A}}$ will denote the submodel of $\mathfrak{A}$ having $C^a_{\mathfrak{A}}$ as its universe. We omit the subscript $\mathfrak{A}$ in $C^a_{\mathfrak{A}}$ and $\mathfrak{C}^a_{\mathfrak{A}}$ when there is no risk of confusion. We call both $C^a_{\mathfrak{A}}$ and $\mathfrak{C}^a_{\mathfrak{A}}$ the component of a in $\mathfrak{A}$. In the Lemma 5.1 it will be convenient to use the following notation:

$$\bar{C}^a_{\mathfrak{A}} = C^a_{\mathfrak{A}} \cap [a|, \text{ and } \underline{C}^a_{\mathfrak{A}} = C^a_{\mathfrak{A}} \cap |a].$$

LEMMA 5.1. *Let $\mathfrak{A}$ be* SA; *then for every $a, b \in A$ either $C^a \cap C^b = \varnothing$ or $C^a = C^b$.*

PROOF. (i). If $b \in \bar{C}^a$ then $\bar{C}^b \supseteq \bar{C}^a \cap [b|$. If not, let $\phi(x, v_0)$ be such that $|\mathfrak{A}|_{\phi,a}$ is convex bounded with minimum a, and such that there is a $d \in |\mathfrak{A}|_{\phi,a}$ such that $d > \bar{C}^b$. By Corollary 2.4 there is a testing formula $\phi^*(x, v_0)$ such that

$$|\mathfrak{A}|_{\phi^*,b} = |\mathfrak{A}|_{\phi,a} \cap [b|$$

hence $d \in |\mathfrak{A}|_{\phi^*,b} \subseteq \bar{C}^b$, contrary to the choice of d. Thus $\bar{C}^b \supseteq \bar{C}^a \cap [b|$.

(ii) If $b \in \bar{C}^a$ then $\bar{C}^b \subseteq \bar{C}^a \cap [b|$. Suppose (ii) is not true. Let $\phi(x, v_0)$ be such that $a, b \in |\mathfrak{A}|_{\phi,a} \subseteq \bar{C}^a$ and $\mathfrak{A}_{\phi,a} \Subset \mathfrak{A}$. Since $\bar{C}^b \supsetneqq \bar{C}^a \cap [b|$ there is $\psi(x, v_0)$ such that $b \in |\mathfrak{A}|_{\psi,b} \leqq \bar{C}^b$ and $|\mathfrak{A}|_{\psi,b} \supsetneqq \bar{C}^a \cap [b|$.

We may assume that for every $c \in A$, $|\mathfrak{A}|_{\psi,c}$ is convex, bounded, and has c as its minimum. Let $\chi(x, v_0) \equiv \phi(x, v_0) \vee \exists y(\phi(x, y) \wedge \psi(y, v_0))$. $|\mathfrak{A}|_{\chi,a}$ is convex, $\min(|\mathfrak{A}|_{\chi,a}) = a$, and $|\mathfrak{A}|_{\chi,a} \supsetneqq \bar{C}^a$. Thus $|\mathfrak{A}|_{\chi,a} = [a|$, and therefore $\mathfrak{A} \models \forall y(y > a \rightarrow \chi(a, y))$. We define $P = \{\sim\psi(d, v_0) \mid d \in |\mathfrak{A}|_{\phi,a}\} \cup \{v_0 > a\}$. Since $|\mathfrak{A}|_{\psi,c}$ is bounded from above for every $c \in A$, P is finitely satisfiable in $\mathfrak{A}$. So there is $\mathfrak{B}$ such that $\mathfrak{A} + \mathfrak{B} \succ \mathfrak{A}$ and $\mathfrak{A} + \mathfrak{B} \models P[c]$ for some $c \in B$. But then $\mathfrak{A} + \mathfrak{B} \models a < c \wedge \forall y(\phi(a, y) \rightarrow \sim\psi(y, c))$, since $|\mathfrak{A}|_{\phi,a}$ is bounded $\mathfrak{A} + \mathfrak{B} \models \sim\phi[a, c]$. Hence $\mathfrak{A} + \mathfrak{B} \models a < c \wedge \sim\chi(a, c)$ and therefore $\mathfrak{A} \models \exists y(a < y \wedge \sim\chi(a, y))$. We arrive at a contradiction. Thus (ii) is true.

(iii) If $b \in \bar{C}^a$ then $a \in \underline{C}^b$. We may assume that $\bar{C}_a$ is bounded from above. Suppose by way of contradiction $b \in \bar{C}^a$ but $a \notin \underline{C}^b$. Let $\phi(x, v_0)$ be such that $|\mathfrak{A}|_{\phi,a}$ is bounded and convex with minimum a and $|\mathfrak{A}|_{\phi,a} \ni b$. We may assume that $|\mathfrak{A}|_{\phi,c}$ is bounded convex with minimum c for every $c \in A$. Then for every $d \in A$ there exists $c < d$ such that $\mathfrak{A} \models \phi[c, b]$. Otherwise, let $\chi \equiv x \geqq v_0 \wedge \exists y(y \leqq v_0 \wedge \phi(y, x))$. Then $|\mathfrak{A}|_{\chi,b}$ is convex bounded with maximum b and contains a; thus $|\mathfrak{A}|_{\chi,b} \supsetneqq \underline{C}^b$ and this is a contradiction. Thus $\mathfrak{A} \models \forall y \exists z(z < y \wedge \phi(z, b))$. Since $\mathfrak{A}$ is SA and $\bar{C}^a$ is bounded there is $b' > \bar{C}^a$ such that $\mathfrak{A} \models \forall y \exists z(z < y \wedge \phi(z, b'))$. Hence there is $a' < a$ such that $\mathfrak{A} \models \phi[a', b']$. Thus $\bar{C}^{a'} \supseteq |\mathfrak{A}|_{\phi,a'} \supsetneqq \bar{C}^a$ which contradicts (i), and (iii) is proved.

If we interchange $\bar{C}$ with $\underline{C}$ in (i), (ii), (iii), clearly we obtain true statements. It is now easy to deduce that if $C^a \cap C^b \neq \varnothing$ then $C^a = C^b$. Q.E.D.

LEMMA 5.2. *Let $\mathfrak{A}$ be* SA, *$\{\mathfrak{C}^{a_i}\}_{i \in I}$ be distinct components in $\mathfrak{A}$ and for every $i \in I$, $\mathfrak{C}_i \prec \mathfrak{C}^{a_i}$. Let $\mathfrak{A}'$ be the submodel of $\mathfrak{A}$ having $(A - \bigcup_{i \in I} C^{a_i}) \cup (\bigcup_{i \in I} C_i)$ as its universe; then for every $i \in I$, C_i is a component in $\mathfrak{A}'$.*

PROOF. $\mathfrak{A}' \prec \mathfrak{A}$, thus if $a \in A'$ then $|\mathfrak{A}|_{\phi,a}$ is convex and bounded iff $|\mathfrak{A}'|_{\phi,a}$ is convex and bounded. It now follows easily that C_i is a component in $\mathfrak{A}'$. We skip the easy proof of the following lemma.

LEMMA 5.3. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be* SA. *Then*

(i) *if f is an isomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$ then for every $a \in A$, $f(C^a_{\mathfrak{A}}) = C^{f(a)}_{\mathfrak{B}}$.*

(ii) *if $\mathfrak{A} \prec \mathfrak{B}$ and $a \in A$ then $\mathfrak{C}^a_{\mathfrak{A}} \prec \mathfrak{C}^a_{\mathfrak{B}}$.*

(iii) *if* $\bar{a} \in A^n$, $\bar{b} \in B^n$, $(\mathfrak{A}, \bar{a}) \equiv (\mathfrak{B}, \bar{b})$, *and the elements of* $\bar{a}$ *are all in the same component* $\mathfrak{C}$, *then the elements of* $\bar{b}$ *are all in the same component, say* $\mathfrak{D}$, *and* $(\mathfrak{C}, \bar{a}) \equiv (\mathfrak{D}, \bar{b})$.

LEMMA 5.4. *Let* $\mathfrak{A}$ *be* SA, $T_{\mathfrak{A}} = T$, $S_1(T)$ *be finite; then*:

(i) *if* $\mathfrak{C}$ *is a coponent in* $\mathfrak{A}$, *then* $\| S_1(T_{\mathfrak{C}}) \| \leqq \| S_1(T) \|$.

(ii) *exactly one of the following alternatives holds*: (a) *For every* $a \in A$, $\mathfrak{C}^a \prec \mathfrak{A}$. (b) *For every* $a \in A$, C^a *is definable over* a. *There is no first or last component. If* $C^{a_1} < C^{a_2}$ *and* $P \in S_1(T)$ *then there is* $a \in A$ *such that* $P(a, \mathfrak{A}) = P$ *and* $C^{a_1} < C^a < C^{a_2}$.

PROOF. By Lemma 5.3(iii), (i) is easy.

(ii). If $C^a = A$ for some $a \in A$ then (a) holds. It is easily seen that for any $\mathfrak{B}$ if $P(b, \mathfrak{B})$ is isolated in $S_1(T_{\mathfrak{B}})$ and $C^b_{\mathfrak{B}}$ is definable over b, then $C^b_{\mathfrak{B}}$ is definable over every element of $C^b_{\mathfrak{B}}$. Suppose $C^a \neq A$ and C^a is not definable over a for some $a \in A$; then clearly C^a is not definable over any of its members. We first show that $\mathfrak{C}^a \prec \mathfrak{A}$. It suffices to show that for every $b \in C^a$ and for every ϕ, if $\mathfrak{A} \models \exists x \phi(b, x)$ then there is $c \in C^a$ such that $\mathfrak{A} \models \phi[b, c]$. Suppose this is not true. We may assume that $\mathfrak{A} \models \phi[b, d], d > \bar{C}^a$ and for no $c \in C^a$, $\mathfrak{A} \models \phi[b, c]$; then clearly $\bar{C}^b$ is definable over b. Let α generate $P(b, \mathfrak{A})$; then

$$\chi(b, v_0) \equiv v_0 \leqq b \wedge \forall y((\alpha(y) \wedge \exists z(\phi(y, z) \wedge z \leqq b)) \rightarrow (y < v_0))$$

defines $\underline{C}^b$ over b so C^b is definable over b and this is a contradiction. We conclude that $\mathfrak{C}^a \prec \mathfrak{A}$. Since $S_1(T)$ is finite, $P(C^a, \mathfrak{A}) = S_1(T)$, by Lemma 5.3(iii), for every component of $\mathfrak{A}, \mathfrak{C}, \mathfrak{C} \equiv \mathfrak{C}^a$; hence $\mathfrak{A}$ is the sum of elementarily equivalent SA models, hence $\mathfrak{A} \succ \mathfrak{C}$ for every component $\mathfrak{C}$ of $\mathfrak{A}$. We showed that if for some $a \in A$, C^a is not definable over a then (a) holds.

Suppose that for every $a \in A$, $C^a \neq A$ and C^a is definable over a. Suppose $C^a < C^b$ and there is $P \in S_1(T)$ which is realized by no element c such that $C^a < c < C^b$. Let $\psi(b, v_0)$ define $\underline{C}^b$ and $\phi(a, v_0)$ define $\bar{C}^a$. Let α generate P and β generate $P(b, \mathfrak{A})$. Let $\chi(a, v_0) \equiv a \leqq v_0 \wedge \exists x(v_0 \leqq x \wedge \beta(x) \wedge \forall y((a \leqq y \leqq x \wedge \sim \psi(x, y) \wedge \sim \phi(a, y)) \rightarrow \sim \alpha(y)))$; then $|\mathfrak{A}|_{\chi, a} = \{c \mid c \in [a, b']$ for some b' such that $P(b', \mathfrak{A}) = P(b, \mathfrak{A})$ and there is no element between $\bar{C}^a$ and $\underline{C}^{b'}$ realizing $P\}$. So $|\mathfrak{A}|_{\chi, a}$ is convex and bounded and $|\mathfrak{A}|_{\chi, a} \supsetneqq \bar{C}^a$ which is impossible. It remains to show that there is no first or last component in $\mathfrak{A}$. If C^a is the last component in $\mathfrak{A}$ then it is definable in $\mathfrak{A}$, so $C^a = A$, in contradiction to our assumption. Thus ii(b) holds. Q.E.D.

LEMMA 5.5. *If* $\mathfrak{A}$ *is* SA, $\mathfrak{A} = \mathfrak{C}_{\mathfrak{A}}^{a'}$ *for some* $a' \in A$, *and* $S_1(T_{\mathfrak{A}})$ *is finite then for every* $a \in A$, *if* $P = P(a, \mathfrak{A})$ *then*:

(i) *there is* $b > a$ *such that* b *realizes* P, $P([a,b], \mathfrak{A}) = S_1(T_{\mathfrak{A}})$ *and for every* $c \in (a,b)$ *which realizes* P *either* $P([a,c], \mathfrak{A}) \neq S_1(T_{\mathfrak{A}})$ *or* $P([c,b], \mathfrak{A}) \neq S_1(T_{\mathfrak{A}})$.

(ii) *there is* $b < a$ *with similar properties.*

PROOF. (i). Suppose by way of contradiction there is no such b. Let $b_0 > a$ be such that $P([a,b_0], \mathfrak{A}) = S_1(T_{\mathfrak{A}})$ and $P(b_0, \mathfrak{A}) = P$. By our assumption there is b_1 which realizes P such that $a < b_1 < b_0$ and

$$P([a,b_1], \mathfrak{A}) = P([b_1,b_0], \mathfrak{A}) = S_1(T_{\mathfrak{A}}).$$

We continue this construction as follows:

$$P([a,b_{i+1}], \mathfrak{A}) = P([b_{i+1},b_i], \mathfrak{A}) = S_1(T_{\mathfrak{A}})$$

for every $i \in \omega$. Let $|\mathfrak{A}|_{\chi,a}$ be convex and bounded $|\mathfrak{A}|_{\chi,a} \ni b_0$. Let b realize P and $b > |\mathfrak{A}|_{\chi,a}$. Hence $b > b_0$. By Corollary 3.9, $P(\langle a,b_0\rangle, \mathfrak{A}) = P(\langle a,b\rangle, \mathfrak{A})$. But this is impossible since $\mathfrak{A} \models \chi[a,b_0]$ but $\mathfrak{A} \models \sim \chi[a,b]$. Thus the lemma is proved.

LEMMA 5.6. *Let* $S_1(T_{\mathfrak{A}})$ *be finite,* $b_1 < b_2 < b_3$ *be elements of* A, $P(b_1, \mathfrak{A}) = P(b_2, \mathfrak{A}) = P$, *and* $P([b_1, b_3], \mathfrak{A}) = S_1(T_{\mathfrak{A}})$. *Then either* $P([b_1,b_2], \mathfrak{A}) = S_1(T_{\mathfrak{A}})$ *or* $P([b_2,b_3], \mathfrak{A}) = S_1(T_{\mathfrak{A}})$.

PROOF. For $i = 1,2$ let $D_i = \bigcup \{[b_i,d] \mid b_i \leq d \text{ and } P([b_i,d], \mathfrak{A}) \neq S_1(T_{\mathfrak{A}})\}$. Clearly since $S_1(T_{\mathfrak{A}})$ is finite then $P(D_1, \mathfrak{A}) = P(D_2, \mathfrak{A}) \neq S_1(T_{\mathfrak{A}})$. If the lemma is not true then $[b_i, b_{i+1}] \subseteq D_i$ for $i = 1,2$, thus $P([b_1, b_3], \mathfrak{A}) \subseteq P(D_1 \cup D_2, \mathfrak{A}) \neq S_1(T_{\mathfrak{A}})$ in contradiction to our assumption on $[b_1, b_3]$. Hence the lemma is true.

LEMMA 5.7. *Let* $\mathfrak{A}$ *be a model in any language. If* $P(\bar{a}, \mathfrak{A}) = P(\bar{b}, \mathfrak{A})$ *and* $\bar{a} \in (|\mathfrak{A}|_{\phi,\bar{a}})^n$, *then* $\bar{b} \in (|\mathfrak{A}|_{\phi,\bar{b}})^n$ *and* $(\mathfrak{A}_{\phi,\bar{a}}, \bar{a}) \equiv (\mathfrak{A}_{\phi,\bar{b}}, \bar{b})$.

PROOF. Trivial.

DEFINTION 5.8. We define inductively the class of models $\mathscr{S}_i$. $\mathscr{S}_0$ is the class of all models in the language L containing a single element. $\mathfrak{A} \in \mathscr{S}_{i+1}$ iff one of the following conditions holds:

(i) $\mathfrak{A} \cong \mathfrak{A}_1 + \mathfrak{A}_2$ and $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathscr{S}_i$.

(ii) $\mathfrak{A} \cong \sum_{j \in J} \mathfrak{A}_j$, for every $j \in J$ $\mathfrak{A}_j \in \mathscr{S}_i$, for every $j_1, j_2 \in J$, $\mathfrak{A}_{j_1} \equiv \mathfrak{A}_{j_2}$, $\langle J, < \rangle \equiv \mathbb{Z}$.

(iii) $\mathfrak{A} \cong \sum_{j \in J} \mathfrak{A}_j$, for every $j \in J$, $\mathfrak{A}_j \in \mathscr{S}_i$; there are $j_1, \cdots, j_r$ such that for every $j \in J$, $\mathfrak{A}_j \equiv \mathfrak{A}_{j_k}$. For some k, $\langle J, < \rangle \equiv \mathbb{Q}$, and for every $k, 1 \leqq k \leqq r$, $\{j \mid \mathfrak{A}_j \equiv \mathfrak{A}_{j_k}\}$ is dense in $\langle J, < \rangle$.

(iv) $\mathfrak{A} \in \mathscr{S}_i$.

Let $\mathscr{S} = \bigcup_{i \in \omega} \mathscr{S}_i$.

THEOREM 5.9. *Let $\mathfrak{A}$ be a model in the language L; then $S_1(T_{\mathfrak{A}})$ is finite if and only if $\mathfrak{A} \in \mathscr{S}$.*

PROOF. We leave it to the reader to prove that, if $\mathfrak{A} \in \mathscr{S}$ then $S_1(T_{\mathfrak{A}})$ is finite.

We show that if $\| S_1(T_{\mathfrak{A}}) \| \leqq n$ then $\mathfrak{A} \in \mathscr{S}_{2n-1}$. It is easy to see that this is true for $n = 1$. Suppose $\| S_1(T_{\mathfrak{A}}) \| = n + 1$. We first consider the case when $\mathfrak{A}$ is not SA: $\mathfrak{A} = \mathfrak{A}_\phi + \mathfrak{A}_{\sim\phi}$ and $\| S_1(T_{\mathfrak{A}_\phi}) \|$, $\| S_1(T_{\mathfrak{A}_{\sim\phi}}) \| \leqq n$. By the induction hypothesis $\mathfrak{A}_\phi, \mathfrak{A}_{\sim\phi} \in \mathscr{S}_{2n-1}$ so $\mathfrak{A} \in \mathscr{S}_{2n}$.

Suppose that $\mathfrak{A}$ fulfills Lemma 5.4(ii:b). We show that for every $a \in A$, $\mathfrak{C}^a \in \mathscr{S}_{2n}$. If $\| S_1(T_{\mathfrak{C}^a}) \| \leqq n$ then this is true by the induction hypothesis. Suppose $\| S_1(T_{\mathfrak{C}^a}) \| = n + 1$ for some $a \in A$. Hence for every $b \in A, \mathfrak{C}^a \equiv \mathfrak{C}^b$; thus if $\mathfrak{C}^a$ is SA then $\mathfrak{C}^a \prec \mathfrak{A}$ which is impossible. Thus $\mathfrak{C}^a$ is not SA; we showed in the first part of the proof that if it is so then $\mathfrak{C}^a \in \mathscr{S}_{2n}$. Let $J \subseteq A$ contain a single representative from every component of $\mathfrak{A}$. It is easy to see that the decomposition $\mathfrak{A} = \sum_{a \in J} \mathfrak{C}^a$ fulfills all the conditions set in Definition 5.8 (iii). We conclude that $\mathfrak{A} \in \mathscr{S}_{2n+1}$.

Suppose $\mathfrak{A}$ fulfills Lemma 5.4(ii, a). Without loss of generality, we may assume that $\mathfrak{A}$ consists of a single component. Let $P \in S_1(T_{\mathfrak{A}})$, let $\{a_z\}_{z \in \mathbb{Z}}$ be a sequence of elements in $\mathfrak{A}$ realizing P. Let $z \to a_z$ be an order isomorphism, and for every $z \in \mathbb{Z}$, $P([a_z, a_{z+1}], \mathfrak{A}) = S_1(T_{\mathfrak{A}})$, and there is no $a \in (a_z, a_{z+1})$ which realizes P such that $P([a_z, a], \mathfrak{A}) = P([a, a_{z+1}], \mathfrak{A}) = S_1(T_{\mathfrak{A}})$. The existence of such a sequence was assured in Lemma 5.5. Since $\mathfrak{A}$ consists of a single component by Corollary 3.9, $\{a_z\}_{z \in \mathbb{Z}}$ is unbounded from below and unbounded from above. Let $z \in \mathbb{Z}$; we define $A_z = \{a \mid a \geqq a_z$. There is $b \geqq a$ which realizes P, and $P([a_z, b], \mathfrak{A}] \neq S_1(T_{\mathfrak{A}})$; or $a \leqq a_z$ and there is $b \leqq a$ which realizes P and $P([b, a_z], \mathfrak{A}) \neq S_1(T_{\mathfrak{A}})\}$.

There is Q_z which is not realized in A_z, for otherwise there are b_1, b_2 which realize P such that $b_1 \leqq a_z \leqq b_2$ and

$$P([b_1, a_z], \mathfrak{A}) \neq S_1(T_{\mathfrak{A}}) \neq P([a_z, b_2], \mathfrak{A})$$

but $P([b_1, b_2], \mathfrak{A}) = S_1(T_{\mathfrak{A}})$. This contradicts Lemma 5.6. Since A_z is definable over a_z and Q_z is isolated, we may assume that $Q_{z_1} = Q_{z_2}$ for every z_1 and z_2. Let $Q_z = Q$. Let $\phi(v_0, x)$ have the following meaning: there is some y which realizes P, $v_0 \in \operatorname{conv}(\{x, y\})$ and $P(\operatorname{conv}(\{x, y\}), \mathfrak{A}) \neq S_1(T_{\mathfrak{A}})$. Using the properties of $\{a_z\}_{z \in \mathbb{Z}}$ and Lemma 5.6 it can be shown that for every z and for every $a \in A_z$, $A_z = |\mathfrak{A}|_{\phi,a}$. By Lemma 5.7 if $b_1, b_2 \in A_z$ and $P(b_1, \mathfrak{A}) = P(b_2, \mathfrak{A})$ then $P(b_1, \mathfrak{A}_z) = P(b_2, \mathfrak{A}_z)$, so since $Q \notin P(A_z, \mathfrak{A})$, $\| S_1(T_{\mathfrak{A}_z}) \| < \| S_1(T_{\mathfrak{A}}) \|$. Moreover, for every z_1, z_2, $\mathfrak{A}_{z_1} \equiv \mathfrak{A}_{z_2}$. Let $B_z = \{a \mid A_z < a < A_{z+1}\}$. Since $Q \in P([a_z, a_{z+1}], \mathfrak{A}) - P(A_z \cup A_{z+1}, \mathfrak{A})$, $B_z \neq \varnothing$. It is easy to see that there is $\psi(v_0, x)$ such that for every z and for every $b \in B_z$, $B_z = |\mathfrak{A}|_{\psi,b}$. Since $P \notin P(B_z, \mathfrak{A})$ again we obtain that $\| S_1(T_{\mathfrak{B}_z}) \| < \| S_1(T_{\mathfrak{A}}) \|$ and $\mathfrak{B}_{z_1} \equiv \mathfrak{B}_{z_2}$ for every z_1 and z_2. By our induction hypothesis $\mathfrak{A}_z, \mathfrak{B}_z \in \mathscr{S}_{2n-1}$ hence $\mathfrak{A}_z + \mathfrak{B}_z \in \mathscr{S}_{2n}$. $\mathfrak{A} = \sum_{z \in \mathbb{Z}} (\mathfrak{A}_z + \mathfrak{B}_z)$ and this decomposition fulfills all the conditions of Definition 5.8 (ii), hence $\mathfrak{A} \in \mathscr{S}_{2n+1}$. Q.E.D.

COROLLARY 5.10. *For every* $n < \omega$, $\{T \mid \| S_1(T) \| \leqq n\}$ *is finite.*

PROOF. We showed that

$$\{T \mid \| S_1(T) \| \leqq n\} \subseteq \{T \mid T \text{ has a model in } \mathscr{S}_{2n-1}\};$$

since our language L is finite $\{T \mid T \text{ has a model in } \mathscr{S}_0\}$ is finite. A straightforward induction shows that $\{T \mid T \text{ has a model in } \mathscr{S}_n\}$ is finite; hence the corollary is proved.

COROLLARY 5.11. *If $L(T)$ is finite and $S_1(T)$ is finite then T has a finite axiomatization.*

PROOF. We follow the proof of Theorem 5.9. If $\|S_1(T)\| = 1$ then, clearly, T has a finite axiomatization. If $\| S_1(T_{\mathfrak{A}}) \| = n + 1$, $\mathfrak{A} = \mathfrak{A}_\phi + \mathfrak{A}_{\sim\phi}$ we construct a finite axiomatization for $\mathfrak{A}$ by means of the axiomatizations for $\mathfrak{A}_\phi$ and $\mathfrak{A}_{\sim\phi}$. The same can be done also in the other cases considered in Theorem 5.9.

We shall say that T is ω-categorical if any two models of T of cardinality $\leqq \omega$ are isomorphic. $\mathfrak{A}$ is said to be ω-categorical if $T_{\mathfrak{A}}$ is ω-categorical.

DEFINITION. We define inductively the class of models $\mathscr{C}_i$. $\mathscr{C}_0 = \mathscr{S}_0$. $\mathfrak{A} \in \mathscr{C}_{i+1}$ iff $\mathfrak{A} \in \mathscr{C}_i$ or $\mathfrak{A}$ decomposes as in Definition 5.8 (i) or (iii). Let $\mathscr{C} = \bigcup_{i \in \omega} \mathscr{C}_i$.

Theorem 5.12 is due to Rosenstein [6].

THEOREM 5.12. $\mathfrak{A}$ *is* ω*-categorial iff* $\mathfrak{A} \in \mathscr{C}$.

We shall need the following lemma:

LEMMA 5.13. *If* $\mathfrak{A} \in \mathscr{C}_n$ *and* $\mathfrak{B} \Subset \mathfrak{A}$ *then* $\mathfrak{B} \in \mathscr{C}_{2n}$.

The proof is easy.

Most of the results of this section hold, possibly with trivial corrections, even if $L(T)$ is infinite. We formulate some of these results for further application.

LEMMA 5.14. *Let L be finite or infinite, then*:

(i) *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are* ω*-categorical models in the language L so is* $\mathfrak{A}+\mathfrak{B}$.

(ii) *If* $\mathfrak{A} = \sum_{j\in J}\mathfrak{A}_j$, $\langle J,<\rangle \equiv \mathbb{Q}$, *there are* $\mathfrak{A}^1,\cdots,\mathfrak{A}^k$ *such that the sets* $J_i = \{j \mid \mathfrak{A}_j \equiv \mathfrak{A}^i\}$ *are dense in* J, $\bigcup_{i=1}^k J_i = J$ *and each* $\mathfrak{A}^i$ *is* ω*-categorical then so is* $\mathfrak{A}$.

LEMMA 5.15. *Let* $\mathfrak{A}$ *be an* ω*-categorical model in the language L (L may be finite or infinite); then*

(i) *If* $\mathfrak{B} \Subset \mathfrak{A}$ *then* $\mathfrak{B}$ *is* ω*-categorical.*

(ii) $\{T_{\mathfrak{B}} \mid \mathfrak{B} \Subset \mathfrak{A}\}$ *is finite.*

6. The number of countable models of a theory

In this section we shall find the number of nonisomorphic countable models of a theory T. In the first part of the section we shall discuss this problem in the case when the models of T are SA. In this case our result does not depend on whether $L(T)$ is finite or countable. However, if the models of T are not SA, we shall obtain different results for $L(T)$ finite and countable. It is Theorem 6.12 which is true when $L(T)$ is finite but not otherwise.

LEMMA 6.1. *Let* $\mathfrak{A}_1$ *be* SA, $\mathfrak{A}_1 \equiv \mathfrak{A}_2 \equiv \mathfrak{A}_3$, *and* $\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_2 + \mathfrak{A}_3$. *Let* $d \in A_2$ *and let* $\mathfrak{B} = \mathfrak{A}_1 + \mathfrak{C}^d_{\mathfrak{A}_2} + \mathfrak{A}_3$; *then* $\mathfrak{B} \prec \mathfrak{A}$.

PROOF. It suffices to show that for every $c \in C^d_{\mathfrak{A}_2}$, for every $b \in |\mathfrak{A}_1| \cup |\mathfrak{A}_3|$ and for every ϕ if there is $a \in A_2$ such that $\mathfrak{A} \models \phi[b,a,c]$, then there is $a' \in B$ such that $\mathfrak{A} \models \phi[b,a',c]$. Without loss of generality, we may assume that $b \in A_1$, $c \in C^d_{\mathfrak{A}_2}$, $a \in A_2 - C^d_{\mathfrak{A}_2}$, $b < a < c$, and $\mathfrak{A} \models \phi[b,a,c]$. Suppose by way of contradiction there is no $a' \in A_1$ such that $\mathfrak{A} \models \phi[b,a',c]$. Let $\phi^*(v_0,v_1)$ be the testing formula of $\phi(v_0,v_1,c)$ in $\mathfrak{A}_1$; then $\mathfrak{A}_1 \models \sim \exists u\phi^*(b,u)$. Since $\mathfrak{A}_1 \prec \mathfrak{A}$, $\mathfrak{A} \models \sim \exists u\phi^*(b,u) \wedge \phi(b,a,c)$. The formula

$$\chi(v_0,c) \equiv v_0 \leqq c \wedge \exists x\ \exists y(x \leqq v_0 \wedge y \leqq v_0 \wedge \sim \exists u\, \phi^*(x,u) \wedge \phi(x,y,c))$$

defines over c a convex set and $|\mathfrak{A}|_{\chi,c} \supsetneq C^d_{\mathfrak{A}_2} \cap |\mathfrak{A},c]$.

Since $C^d_{\mathfrak{A}_2} = C^c_{\mathfrak{A}}$, $|\mathfrak{A}|_{\chi,c} \supseteq \underline{C}^c_{\mathfrak{A}}$, hence $|\mathfrak{A}|_{\chi,c} = |\mathfrak{A},c]$.

Let $d' \in A_1$; since $\mathfrak{A} \models \chi[d',c]$, there are $b', a' \leqq d'$ such that $\mathfrak{A} \models \sim\exists u\phi^*(b',u) \wedge \phi[b',a',c]$, and so $\mathfrak{A}_1 \models \phi^*[b',a']$. Since $\mathfrak{A}_1 \prec \mathfrak{A}$, $\mathfrak{A} \models \phi^*[b',a']$, but this is impossible. So there exists $a' \in A_1$, such that $\mathfrak{A} \models \phi[b, a', c]$, and the lemma is proved.

LEMMA 6.2. *If* $\| S_1(T) \| \geqq \omega$ *and the models of* T *are* SA, *then* T *has* 2^ω *nonisomorphic countable models.*

PROOF. Let P be a nonisolated type of $S_1(T)$. Let $\mathfrak{A}_1$ be a countable model of T in which P is realized, and let $\mathfrak{A}_2$ be a countable model of T omitting P. Let $\mathfrak{B} = \mathfrak{A}_2 + \mathfrak{C}^a_{\mathfrak{A}_1} + \mathfrak{A}_2$ where $P(a, \mathfrak{A}_1) = P$. Since $\mathfrak{B} \prec \mathfrak{A}_2 + \mathfrak{A}_1 + \mathfrak{A}_2 \succ \mathfrak{A}_1$, $P(a, \mathfrak{B}) = P$ and $\mathfrak{C}^a_{\mathfrak{B}} = \mathfrak{C}^a_{\mathfrak{A}_1}$. If $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are nonisomorphic ordered sets then $\mathfrak{B} \cdot \mathfrak{D}_1$ and $\mathfrak{B} \cdot \mathfrak{D}_2$ are nonisomorphic, for the set of components of $\mathfrak{B} \cdot \mathfrak{D}_1$ in which P is realized has the order type of $\mathfrak{D}_1$ and the same set of components in $\mathfrak{B} \cdot \mathfrak{D}_2$ has the order type of $\mathfrak{D}_2$. Since there are 2^ω countable order types we obtain 2^ω nonisomorphic models of the form $\mathfrak{B} \cdot \mathfrak{D}$ where $\mathfrak{D}$ is a countable ordered set, and the lemma is proved.

We obtain a similar result when $S_1(T)$ is finite.

LEMMA 6.3. *If* $S_1(T)$ *is finite, then either* T *is* ω*-categorical or* T *has* 2^ω *nonisomorphic countable models.*

PROOF. Without loss of generality, we can assume that $L(T)$ is finite. We prove the theorem by induction on $\| S_1(T) \|$. If $\| S_1(T) \| = 1$ then it is easy to check that T has a model which contains a single point, or T has a model of order type η, or T has a model of order type $\mathbb{Z}$, and each unary predicate in a model of T is either empty or the whole universe. In any of these cases either T is ω-categorical or T has 2^ω nonisomorphic countable models.

Suppose $\| S_1(T) \| = n + 1$. We distinguish between the case when the models of T are not SA, and cases (ii, a) and (ii, b) of Lemma 5.4. In the first case let $\mathfrak{A}$ be a countable model of T; then $\mathfrak{A} = \mathfrak{A}_\phi + \mathfrak{A}_{\sim\phi}$ where $\| S_1(T_{\mathfrak{A}\phi}) \| \leqq n$ and $\| S_1(T_{\mathfrak{A}\sim\phi}) \| \leqq n$. If $\mathfrak{A}_\phi$ and $\mathfrak{A}_{\sim\phi}$ are both ω-categorical then so is $\mathfrak{A}$. If, say, $\mathfrak{A}_\phi$ is not ω-categorical then by the induction hypothesis there is a set of countable models $\{ \mathfrak{B}_\nu \mid \nu < 2^\omega \}$ such that for every ν, $\mathfrak{B}_\nu \equiv \mathfrak{A}_\phi$ and if $\nu_1 \neq \nu_2$ then $\mathfrak{B}_{\nu_1} \not\cong \mathfrak{B}_{\nu_2}$. Let $\mathfrak{A}_\nu = \mathfrak{B}_\nu + \mathfrak{A}_{\sim\phi}$. Obviously $\mathfrak{A}_\nu \equiv \mathfrak{A}$, and by Lemma 2.5 (ii) if $\nu_1 \neq \nu_2$, $\mathfrak{A}_{\nu_1} \not\cong \mathfrak{A}_{\nu_2}$. In case (a) of Lemma 5.4 (ii) T has a countable model $\mathfrak{A}$ consisting of a single component. If $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are nonisomorphic ordered sets, then clearly, $\mathfrak{A} \cdot \mathfrak{D}_1 \not\cong \mathfrak{A} \cdot \mathfrak{D}_2$ and $\mathfrak{A} \cdot \mathfrak{D}_i \equiv \mathfrak{A}$. So obviously T has 2^ω nonisomorphic countable models. In Case (b) of Lemma 5.4 (ii) choose a model $\mathfrak{A}$ of T such that: $\mathfrak{A} = \sum_{r \in Q} \mathfrak{A}_r$, $\mathfrak{A}$ is countable, each $\mathfrak{A}_r$ is a component of $\mathfrak{A}$, and there are $\mathfrak{A}^1, \cdots, \mathfrak{A}^k$ such that $\mathfrak{A}^i \not\equiv \mathfrak{A}^j$ for $i \neq j$, and the sets $Q_i = \{ r \mid \mathfrak{A}_r \cong \mathfrak{A}^i \}$

are dense in $\mathbf{Q}$ and $\bigcup_{i=1}^{k} Q_i = \mathbf{Q}$. We further assume that each $\mathfrak{A}_r$ is definable over any of its elements. If the $\mathfrak{A}_i$'s are all ω-categorical then so is $\mathfrak{A}$. If $\mathfrak{A}^{i_0}$ is not ω-categorical for some i_0, let $T' = T_{\mathfrak{A}^{i_0}}$, then either $\| S_1(T') \| \leqq n$ or $\| S_1(T') \| = n+1$ and $\mathfrak{A}^{i_0}$ is not SA. Thus by the induction hypothesis or by the same argument as in the first case, T' has 2^ω nonisomorphic countable models. Let $\{\mathfrak{B}_\nu \mid \nu < 2^\omega\}$ be a set of such models, and define $\mathfrak{A}_\nu = \sum_{r \in Q} \mathfrak{A}'_r$ where $\mathfrak{A}'_r = \mathfrak{A}_r$ if $r \notin Q_{i_0}$ and $\mathfrak{A}'_r = \mathfrak{B}_\nu$ if $r \in Q_{i_0}$. Then obviously $\mathfrak{A}_\nu$ is a model of T. If $\nu_1 \neq \nu_2$ then $\mathfrak{A}_{\nu_1} \not\cong \mathfrak{A}_{\nu_2}$, for the components of $\mathfrak{A}_{\nu_1}$ which are elementarily equivalent to $\mathfrak{A}^{i_0}$ are not isomorphic to the components of $\mathfrak{A}_{\nu_2}$ which are elementarily equivalent to $\mathfrak{A}^{i_0}$. Hence T has 2^ω nonisomorphic countable models, and the lemma is proved.

LEMMA 6.4. *If for some model $\mathfrak{A}$ of T and some $\Phi \in \mathscr{F}_T$, $\mathfrak{A}_\Phi$ is not ω-categorical, then T has 2^ω nonisomorphic countable models.*

PROOF. Let $\mathfrak{B} \succ \mathfrak{A}$ and $\mathfrak{B}$ be ω-saturated. Let $\mathfrak{D}$ be the submodel of $\mathfrak{B}$ with universe $\mathrm{conv}(|\mathfrak{A}|_\Phi, \mathfrak{B})$. By Theorem 2.10(i), $\mathfrak{D} \succ \mathfrak{A}_\Phi$, hence $\mathfrak{D}$ is not ω-categorical, but $\mathfrak{D} \Subset \mathfrak{B}_\Phi$, thus by Lemma 5.15(i), $\mathfrak{B}_\Phi$ is not ω-categorical. By Lemma 4.8, $\mathfrak{B}_\Phi$ is SA. Hence by Lemmas 6.2 and 6.3 there are 2^ω nonisomorphic countable models elementarily equivalent to $\mathfrak{B}_\Phi$. Let $\{\mathfrak{B}_\nu \mid \nu < 2^\omega\}$ be a set of such models.

Let $\mathfrak{C} \prec \mathfrak{B}$ be countable and

$$\underline{C} = \{c \mid c \in C \text{ and } c < |\mathfrak{B}_\Phi|\},\ \bar{C} = \{c \mid c \in C \text{ and } c > |\mathfrak{B}_\Phi|\};$$

then $\{\underline{\mathfrak{C}} + \mathfrak{B}_\nu + \bar{\mathfrak{C}} \mid \nu < 2^\omega\}$ is a set of 2^ω nonisomorphic countable models of T.

LEMMA 6.5. *Let $\Phi \in \mathscr{F}_T$ and $\mathscr{T} = \{T_{\mathfrak{A}_\Phi} \mid \mathfrak{A}$ is a model of $T\}$; then if every $T' \in \mathscr{T}$ is ω-categorical then $\mathscr{T}$ is finite.*

PROOF. For every $T' \in \mathscr{T}$ let $\mathfrak{A}(T')$ be a model of T such that $\mathfrak{A}(T')_\Phi \models T'$. Let $\mathfrak{B}$ be ω-saturated and $\mathfrak{B} \succ \mathfrak{A}(T')$ for every $T' \in \mathscr{T}$; then, as in Lemma 6.4, for every $T' \in \mathscr{T}$, $\mathfrak{B}_\Phi \supseteq \mathfrak{D}_{T'} \succ \mathfrak{A}(T')_\Phi$ where $D_{T'} = \mathrm{conv}(|\mathfrak{A}(T')|_\Phi, \mathfrak{B})$. Since $\mathfrak{B}_\Phi$ is ω-categorical, by Lemma 5.15(ii), $\mathscr{T}$ is finite.

LEMMA 6.6. *Let $\mathscr{K} \subseteq \mathscr{K}_{\mathfrak{A}}$ be a set of limit kernels, such that for no $K \in \mathscr{K}$, K is an accumulation point of $\mathscr{K}$ relative to $\mathscr{K}_{\mathfrak{A}}$. Let $\mathfrak{B}$ be the submodel of $\mathfrak{A}$ having $A - \bigcup \{K \mid K \in \mathscr{K}\}$ as its universe; then $\mathfrak{B} \prec \mathfrak{A}$.*

PROOF. Suppose it is not true that $\mathfrak{B} \prec \mathfrak{A}$; then there is $\bar{b} \in B^k$ and a formula $\phi(\bar{b}, v_0)$ and an element $a \in A - B$ such that $\mathfrak{A} \models \phi[\bar{b}, a]$ but for no $b \in B$, $\mathfrak{A} \models \phi[\bar{b}, b]$. Let $a \in K \in \mathscr{K}$; by our assumption on $\mathscr{K}$ there is $\psi(v_0)$ such that

$|\mathfrak{A}|_\psi \supseteq K$, $\mathfrak{A}_\psi \subset \mathfrak{A}$, $|\mathfrak{A}|_\psi \cap K' = \emptyset$ for every $K' \in \mathscr{K} - \{K\}$, and $|\mathfrak{A}|_\psi$ does not intersect $\bar{b}$. By Corollary 2.4 there is $\phi^*(v_0)$ such that for every $a' \in A$, $\mathfrak{A} \models \phi^*[a']$ iff $a' \in |\mathfrak{A}|_\psi$ and $\mathfrak{A} \models \phi[\bar{b}, a']$. Let

$$\alpha(v_0) \equiv \exists x_1 \exists x_2 (x_1 \leqq v_0 \leqq x_2 \wedge \phi^*(x_1) \wedge \phi^*(x_2));$$

then α defines a convex nonempty subset of A. ϕ^* is satisfied by no element outside K, for $|\mathfrak{A}|_\psi - K \subseteq B$, and ϕ is satisfied by an element of B. Thus $|\mathfrak{A}|_\alpha \subseteq K$. But $K \in \mathscr{K}_\mathfrak{A}$ so $|\mathfrak{A}|_\alpha = K$, contradicting the fact that K is a limit kernel. Thus $\mathfrak{B} \prec \mathfrak{A}$, and the lemma is proved.

THEOREM 6.7. *Either T has only a finite number of nonisomorphic countable models or T has 2^ω nonisomorhpic countable models.*

PROOF. By Lemmas 6.2 and 6.3 the theorem is true when the models of T are SA. If for some n, $\| S_n(T) \| = 2^\omega$, trivially T has 2^ω nonisomorphic countable models. Suppose the models of T are not SA and $\| S_n(T) \| \leqq \omega$ for every n. Let $\mathfrak{A}$ be a countable ω-saturated model of T. If there are infinitely many limit kernels in $\mathscr{K}_\mathfrak{A}$, let $\mathscr{K}$ be an infinite subset of $\mathscr{K}_\mathfrak{A}$ consisting of limit kernels, such that for no $K \in \mathscr{K}$, K is an accumulation point of $\mathscr{K}$. For every $\mathscr{K}' \subseteq \mathscr{K}$ let $\mathfrak{A}_{\mathscr{K}'}$ be the submodel of $\mathfrak{A}$ having $A - \bigcup\{K' \mid K' \in \mathscr{K}'\}$ as its universe. By Lemma 6.6 $\mathfrak{A}_{\mathscr{K}'}$ is a model of T for every $\mathscr{K}' \subseteq \mathscr{K}$. Obviously if $\mathscr{K}_1 \neq \mathscr{K}_2$ then $\mathfrak{A}_{\mathscr{K}_1} \ncong \mathfrak{A}_{\mathscr{K}_2}$. Thus we obtain 2^ω nonisomorphic models of T as demanded.

Suppose now that the models of T have only finitely many limit kernels. If there is a model of T, $\mathfrak{A}$, and $\Phi \in \mathscr{F}_T$ such that $\mathfrak{A}_\Phi$ is not ω-categorical, then by Lemma 6.4, T has 2^ω nonisomorphic countable models.

It remains to consider the case where there are only finitely many limit kernels in $\mathscr{F}_T$, and for every $\Phi \in \mathscr{F}_T$, and a model $\mathfrak{A}$ of T, $\mathfrak{A}_\Phi$ is ω-categorical. For every $\Phi \in \mathscr{F}_T$ let $\mathscr{T}_\Phi = \{T_{\mathfrak{A}_\Phi} \mid \mathfrak{A}$ is a model of $T\}$; then by Lemma 6.5 $\mathscr{T}_\Phi$ is finite for every $\Phi \in \mathscr{F}_T$. Further, if Φ is isolated in $\mathscr{F}_T$ then $\| \mathscr{T}_\Phi \| = 1$. Since there are only finitely many limit types in $\mathscr{F}_T$ there are only finitely many $\mathscr{T}_\Phi$'s for which $\| \mathscr{T}_\Phi \| > 1$. It is now obvious that such T has only finitely many nonisomorphic countable models, and the theorem is proved.

REMARK. Indeed for every positive $n \neq 2$ there is T such that T has exactly n nonisomorphic countable models.

The result of Theorem 6.7 can be sharpened when $L(T)$ is finite, namely, the number of nonisomorphic countable models is then either one or 2^ω. The difference arises because if $L(T)$ is finite and Φ is a limit type in $\mathscr{T}_T$ then there is always a model of T, $\mathfrak{A}$ such that $\mathfrak{A}_\Phi$ is not ω-categorical. We already saw in

Lemma 6.4 that if this is the case then T has 2^{ω} nonisomorphic countable models.

LEMMA 6.8. *Let $\mathfrak{A}$ be a model of T, $L(T)$ is finite,*

$$\{\phi_i \mid i \in \omega\} \in \mathscr{F}_T,\ \mathfrak{B}_i \Subset \mathfrak{A}_{\wedge_{j \leqq i}\phi_j},\ \mathfrak{B}_i \Subset_{g_i} \mathfrak{B}_{i+1},$$

$\mathfrak{B}_i$ is ω-categorical, and $\mathfrak{B} = \bigcup_{i\in\omega}(\mathfrak{B}_i, g_i)$; then there is $\mathfrak{A}' \succ \mathfrak{A}$ such that $\mathfrak{B} \Subset \mathfrak{A}'_{\{\phi_i \mid i\in\omega\}}$.

PROOF. Let $\{b_z \mid z \in \mathbb{Z}\} \subseteq B$ be such that $b_{z_1} \leqq b_{z_2}$ for $z_1 \leqq z_2$ and $\bigcup_{z>0}[b_{-z}, b_z] = B$. For every $z > 0$ there is i such that $B_i \supseteq [b_{-z}, b_z]$. Let $\mathfrak{D}_z = (\mathfrak{D}'_z, \langle b_{-z}, b_{-z+1}, \cdots, b_z\rangle)$ where $\mathfrak{D}'_z$ is the submodel of $\mathfrak{B}$ with universe $[b_{-z}, b_z]$. By Lemma 5.13 $\mathfrak{D}_z$ is ω-categorical and by Corollary 5.11 it has a a finite axiomatization, say, ψ'_z. Let ψ_z be the relativization of ψ'_z to the submodel defined by the formula $\tilde{b}_{-z} \leqq v_0 \leqq \tilde{b}_z$, that is, ψ_z says that $[b_{-z}, b_z]$ satisfies ψ'_z. We may assume that $A \cap B = \emptyset$. Let $\Sigma = D(\mathfrak{A}) \cup \{\psi_z \mid z > 0\} \cup \{\phi_i(\tilde{b}_z) \mid i \in \omega\}$ where $D(\mathfrak{A})$ is the complete diagram of $\mathfrak{A}$. It is easily seen that Σ is consistent. Let $\mathfrak{A}'$ be a countable model of Σ and let a_z be the interpretation of $\tilde{b}_z$ in $\mathfrak{A}'$. Let $\mathfrak{B}'$ be the submodel of $\mathfrak{A}'$ having $\mathrm{conv}(\{a_z \mid z \in \mathbb{Z}\})$ as its universe, and $\mathfrak{C}_z$ the submodel of $\mathfrak{A}'$ with universe $[a_{-z}, a_z]$. Since $\mathfrak{C}_z \cong \mathfrak{D}_z$ $\mathfrak{B}' \cong \mathfrak{B}$. Obviously $\mathfrak{A}' \restriction L(T) \succ \mathfrak{A}$ and $\mathfrak{B}' \Subset \mathfrak{A}'_{\{\phi_i \mid i\in\omega\}}$. Q.E.D.

LEMMA 6.9. *Let $\langle A, < \rangle$ be an infinite, partially ordered set. For every $a \in A$ $\{x \mid x < a\}$ is finite. If $a \in A$ we define $h(a) = \max\{n \mid$ there are a_i, $i = 0, \cdots, n$, such that $a_0 < a_1 \cdots < a_n = a\}$. For every n $\{a \mid h(a) \leqq n\}$ is finite. Then there is $B \subseteq A$ of order type ω.*

PROOF. We define a new relation on A. $a <_1 b$ if there is a chain $a_0 < a_1 < \cdots < a_{h(b)} = b$ such that $a = a_i$ for some i, $0 \leqq i < h(b)$. It is easily seen that $<_1$ partially orders A, if $a <_1 b$ then $a < b$, and if we define $h_1(a) = \max\{n \mid \text{there are } a_i, i = 0, \cdots, n, \text{ such that } a_0 <_1 a_1 <_1 \cdots <_1 a_n = a\}$, then for every $a \in A$, $h(a) = h_1(a)$. We define inductively sequences b_i and B_i such that B_i is infinite and for every $b \in B_i$, $b_i <_1 b$. For every $a \in A$ there is $b \leqq_1 a$ such that $h_1(b) = 0$. Since $\{b \mid h_1(b) = 0\}$ is finite there is b_0 such that $h_1(b_0) = 0$ and $\{b \mid b_0 <_1 b\}$ is infinite. Let $B_0 = \{b \mid b_0 <_1 b\}$. Suppose b_i and B_i are already defined. If b is a successor of b_i relative to $<_1$ then $h(b) = h(b_i) + 1$. Thus $\{b \mid b$ is a successor of b_i relative to $<_1\}$ is finite. There is b' which is a successor of b relative to $<_1$ such that $\{b \mid b \in B_i$ and $b' <_1 b\}$ is infinite. Define $b_{i+1} = b'$ and $B_{i+1} = \{b \mid b \in B_i$ and $b_{i+1} <_1 b\}$. The set $B = \{b_i \mid i \in \omega\}$ is as desired.

LEMMA 6.10. *Let L be a finite language, $\mathscr{M}$ be an infinite set of ω-categorical nonisomorphic countable models in the language L such that if $\mathfrak{A} \in \mathscr{M}$ and $\mathfrak{B} \subset \mathfrak{A}$, then there is $\mathfrak{B}' \in \mathscr{M}$ such that $\mathfrak{B}' \cong \mathfrak{B}$; then there are sequences $\{\mathfrak{A}_i\}_{i \in \omega}$ and $\{g\}_{i \in \omega}$ such that for every $i \in \omega$, $\mathfrak{A}_i \in \mathscr{M}$ and $\mathfrak{A}_i \subset_{g_i} \mathfrak{A}_{i+1}$.*

PROOF. We define a partial order $<$ on $\mathscr{M}$: $\mathfrak{B} < \mathfrak{A}$ iff for some n $\mathfrak{B} \in \mathscr{C}_n$, $\mathfrak{B} \notin \mathscr{C}_{n+1}$, $\mathfrak{A} \in \mathscr{C}_{n+1} - \mathscr{C}_n$ and $\mathfrak{B} \subset_g \mathfrak{A}$ for some g. Obviously $<$ is a partial order; if $\mathfrak{A} \in \mathscr{M} \cap \mathscr{C}_n$ then $\{\mathfrak{B} \mid \mathfrak{B} < \mathfrak{A}\} \subseteq \mathscr{M} \cap \mathscr{C}_{n-1}$ and is therefore finite. It is easily seen that for every $\mathfrak{A} \in \mathscr{M}$, $h(\mathfrak{A}) = n$ iff $\mathfrak{A} \in \mathscr{C}_n$; hence $\{\mathfrak{A} \mid h(\mathfrak{A}) = n\}$ is finite. By Lemma 6.9 $\mathscr{M}$ has a subset of order type ω, and this proves the lemma.

LEMMA 6.11. *If L is a finite language and for every $i \in \omega$ $\mathfrak{A}_i$ is an ω-categorical countable model in the language L, $\mathfrak{A}_i \not\cong \mathfrak{A}_j$ for every $i \neq j$ and $\mathfrak{A}_i \subset \mathfrak{A}_{i+1}$ for every $i \in \omega$, then $\bigcup_{i \in \omega} \mathfrak{A}_i$ is not ω-categorical.*

PROOF. Suppose $\bigcup_{i \in \omega} \mathfrak{A}_i \in \mathscr{C}_n$, then by Lemma 5.13 each $\mathfrak{A}_i \in \mathscr{C}_{2n}$, and this is impossible, for there are only finitely many complete theories which have a model in $\mathscr{C}_{2n}$.

THEOREM 6.12. *If $L(T)$ is finite then either T is ω-categorical or T has 2^ω nonisomorphic coutable models.*

PROOF. By Lemmas 6.2 and 6.3 the theorem is true for theories whose models are SA. By the proof of Theorem 6.7, if there are infinitely many limit types in $\mathscr{F}_T$ then T has 2^ω nonisomorphic countable models. Suppose there are only finitely many limit types in $\mathscr{F}_T$. Let $\mathfrak{A}$ be a countable model of T. If $\mathfrak{A}_\Phi$ is not ω-categorical for some isolated Φ in $\mathscr{F}_T$ then, since $\mathfrak{A}_\Phi$ is SA, T has 2^ω nonisomorphic countable models.

Suppose that for every isolated Φ in $\mathscr{F}_T$, $\mathfrak{A}_\Phi$ is ω-categorical. If $\mathscr{K}_\mathfrak{A}$ is finite then T is ω-categorical; otherwise, we may assume that $\mathscr{K}_\mathfrak{A}$ has a convex subset of order type $\omega + 1$. Let this subset be $\{\mathfrak{K}_i\}_{i \in \omega+1}$ and $i \to \mathfrak{K}_i$ be an order isomorphism. Suppose first that there is an infinite subset of ω, M, such that if $i,j \in M$ and $i \neq j$ then $\mathfrak{K}_i \not\cong \mathfrak{K}_j$. By Lemma 6.10, there are sequences $\{v_i\}_{i \in \omega}$, $\{\mathfrak{C}_i\}_{i \in \omega}$ such that $v_i > v_j$ for $i > j$ and $v_i \in M$ for every i, $\mathfrak{C}_i \subset \mathfrak{K}_{v_i}$ and for every i there is g_i such that $\mathfrak{C}_i \subset_{g_i} \mathfrak{C}_{i+1}$. Let $\mathfrak{K}_\omega = \mathfrak{A}_\Phi$ where $\Phi \in \mathscr{F}_T$; then by Lemma 6.8, $\bigcup_{i \in \omega}(\mathfrak{C}_i, g_i) \subset \mathfrak{B}_\Phi$ for some $\mathfrak{B} \equiv \mathfrak{A}$. By Lemma 6.11, $\bigcup_{i \in \omega}(\mathfrak{B}_i, g_i)$ is not ω-categorical, so by Lemma 5.13, $\mathfrak{B}_\Phi$ is not ω-categorical. Hence, by Lemma 6.4, T has 2^ω nonisomorphic countable models.

Suppose that there are models $\mathfrak{C}_1, \cdots, \mathfrak{C}_n$ such that for every $i \in \omega$, $\mathfrak{K}_i \cong \mathfrak{C}_j$ for some j. We define by induction a sequence $\{k_i\}_{i \in \omega}$ such that $1 \leqq k_i \leqq n$ and for every i $\{v \mid \mathfrak{K}_{v+j} \cong \mathfrak{C}_{k_j}, 0 \leqq j \leqq i\} = M_i$ is infinite. It is easy to define k_0. Suppose $k_0, \cdots, k_i$ are already defined. By the induction hypothesis M_i is infinite so there is r, such that $\{v \mid v \in M_i$ and $\mathfrak{K}_{v+i+1} \cong \mathfrak{C}_r\}$ is infinite. Let $k_{i+1} = r$, then k_{i+1} has the demanded properties. Let $\mathfrak{D}_i = \Sigma_{j=0}^{i} \mathfrak{C}_{k_j}$; then $\mathfrak{D}_i$ is ω-categorical since each $\mathfrak{C}_r$ is ω-categorical. If $i > j$ then $\mathfrak{D}_i \supset \mathfrak{D}_j$. For every $i > j$ there are $\mathfrak{D}_i'$ and $\mathfrak{D}_j'$ such that $\mathfrak{A} \supset \mathfrak{D}_i' \supset \mathfrak{D}_j'$, $D_i' \supsetneqq D_j'$, $\mathfrak{D}_i' \cong \mathfrak{D}_i$, and $\mathfrak{D}_j' \cong \mathfrak{D}_j$; D_j' is definable in $\mathfrak{A}$, hence D_j' is definable in $\mathfrak{D}_i'$, so $\| S_1(T_{\mathfrak{D}_i}) \| > \| S_1(T_{\mathfrak{D}_j}) \|$ whence $\mathfrak{D}_i \not\cong \mathfrak{D}_j$. By Lemma 6.11, $\bigcup_{i \in \omega} \mathfrak{D}_i$ is not ω-catgeorical and by Lemma 6.8, $\bigcup_{i \in \omega} \mathfrak{D}_i \in \mathfrak{B}_\Phi$ for some $\mathfrak{B} \equiv \mathfrak{A}$. So by Lemma 6.4 we conclude that there are 2^ω nonisomorphic countable models elementarily equivalent to $\mathfrak{A}$. Q.E.D.

The following theorem can be easily proved by the methods of this section.

THEOREM 6.13. *If T has 2^ω nonisomorphic countable models, then every countable model of T has 2^ω nonisomorphic countable elementary extensions.*

REMARK. We did not succeed in answering the following question, which is a special case of a more general open problem.

Let ϕ be a sentence of $L_{\omega_1,\omega}$ such that $<$ (and equality) are the only nonlogical symbols which occur in ϕ. Suppose all the models of ϕ are linearly ordered by $<$. How many nonisomorphic countable models does ϕ have? We solved this question in the following special case:

THEOREM. *Let T be a (first order) theory of linear order and S a countable set of types (not necessarily complete) of a single element. Then the number of nonisomorphic countable models of T which omit the types of S is either 2^ω or $\leqq \omega$.*

7. The relation between $S_1(T)$ and $S_n(T)$

In this section we investigate the relation between $S_1(T)$ and $S_n(T)$. It turns out that if $S_1(T)$ is small then so is $S_n(T)$. Corollary 7.22 and Theorem 7.27 express this fact. We shall measure the size of $S_n(T)$ not only by its cardinality but also by its Cantor Bendixon rank. We thus need some topological preliminaries.

DEFINITIONS. Let X be a topological space; then $D^1(X)$ denotes the set of accumulation points in X, $D^0(X) = X$, $D^{v+1}(X) = D^1(D^v(X))$, and if δ is a limit ordinal then $D^\delta(X) = \bigcap_{v<\delta} D^v(X)$.

We define the rank of x in X to be ∞ if $x \in D^\nu(X)$ for every ν and to be $\bigcup\{\nu \mid x \in D^\nu(X)\}$ otherwise. We denote the rank of x in X by $R(x,X)$. Clearly, if $R(x,X) = \nu < \infty$ then $x \in D^\nu(X)$.

We define $R(X)$, the rank of X, to be ∞ if there is some $x \in X$ for which $R(x,X) = \infty$ and to be $\bigcup \{R(x,X) \mid x \in X\}$ otherwise.

The proof of the following theorem can be found in [7, p. 170].

THEOREM 7.1. (i) *If X is a countable Hausdorff compact space then there is $\xi < \omega_1$ such that $D^\xi(X) = \emptyset$.*

(ii) *If X is a separable compact Hausdorff space then either $\| X \| \leqq \omega$ or $\| X \| = 2^\omega$.*

The next lemma is well known.

LEMMA 7.2. *A continuous one-to-one function from a compact space to a Hausdorff space is a homeomorphism into the second space.*

From now on we confine ourselves to countable Hausdorff compact spaces; X and Y will denote only such spaces. We list some elementary properties.

LEMMA 7.3. (i) *For every $\nu, D^\nu(X)$ is closed in X.*

(ii) *$R(x,X) \geqq \nu$ iff for every neighbourhood V of x and for every $\xi < \nu$ there is $x' \neq x$ in V such that $R(x',X) \geqq \xi$.*

(iii) *$R(x,X) \leqq \nu$ iff x has a neighbourhood V such that $R(x',X) < \nu$ for every $x' \neq x$ in V.*

(iv) *If $Y \subseteq X$ then $D^\nu(Y) \subseteq D^\nu(X)$.*

LEMMA 7.4. $D^{\nu+\xi}(X) = D^\xi(D^\nu(X))$.

PROOF. The lemma is easily proved by induction on ξ.

LEMMA 7.5. *If X_i are closed subsets of X, $i = 1,\cdots,n$, and $\bigcup_{i=1}^n X_i = X$ then:*

(i) *For every ν, $D^\nu(X) = \bigcup_{i=1}^n D^\nu(X_i)$.*

(ii) *If $x \in X$ then there is i such that $R(x,X) = R(x,X_i)$.*

(iii) $R(X) = \max\{R(X_i) \mid i = 1,\cdots,n\}$.

PROOF. (ii) and (iii) are trivial consequences of (i). So it remains to prove (i). We prove (i) by induction on ν. For $\nu = 0$ there is nothing to prove. Let δ be a limit ordinal and suppose the induction hypothesis is true for every $\nu < \delta$. Let $x \in D^\delta(X)$; then $x \in D^\nu(X)$ for every $\nu < \delta$. By the induction hypothesis there is i_ν such that $x \in D^\nu(X_{i_\nu})$, hence there is some i such that for every $\nu < \delta$, $x \in D^\nu(X_i)$

hence $x \in D^{\delta}(X_i)$. Let $x \in D^{\nu+1}(X)$ and suppose the induction hypothesis is true for ν. Let V be a neighbourhood of x; then there exists $x' \neq x$ in V such that $x' \in D^{\nu}(X)$, thus $x' \in D^{\nu}(X_i)$ for some i. For every neighbourhood V of x there is some i and some $x' \neq x$ in V such that $x' \in D^{\nu}(X_i)$. Thus there is an i such that for every neighbourhood V of x there is $x \neq x'$ in V such that $x' \in D^{\nu}(X_i)$. Since X_i is closed, $x \in X_i$ and thus $x \in D^{\nu+1}(X_i)$.

LEMMA 7.6. *Let* $\mathfrak{A} \supset \mathfrak{B}$, $n \geqq 1$. $C = \{P(\bar{b},\mathfrak{B}) \mid \bar{b} \in B^n\}$. *We define*

$$f: C \to S_n(T_{\mathfrak{A}}):\ \ f(P(\bar{b},\mathfrak{B})) = P(\bar{b},\mathfrak{A});$$

then f is continuous.

PROOF. If ϕ is a formula with its free variables among $v_0, \cdots, v_{n-1}$, denote $V_\phi = \{Q \mid Q \in S_n(T_{\mathfrak{A}})$ and $Q \ni \phi\}$. Then the set of all such V_ϕ's is a basis for $S_n(T)$. Thus it suffices to show that $f^{-1}(V_\phi)$ is open in C. Let ϕ^* be the testing formula for ϕ; then $f^{-1}(V_\phi) = U_{\phi^*} \cap C$ where $U_\psi = \{Q \mid \psi \in Q \in S_n(T_{\mathfrak{B}})\}$. So $f^{-1}(V_\phi)$ i open, and f is continuous.

LEMMA 7.7. *Let $\mathfrak{A}$ be ω-saturated, $a \in A$, and $B = |\mathfrak{A}|_{\Phi,a}$. Then:*

(i) *$P(B,\mathfrak{A})$ is closed in $S_1(T_{\mathfrak{A}})$.*

(ii) *$\{P(\langle a,b\rangle,\mathfrak{A}) \mid b \in B\}$ is closed in $S_2(T_{\mathfrak{A}})$.*

(iii) *If δ is a limit ordinal and for every $\nu < \delta$, $D^{\nu}(S_1(T_{\mathfrak{A}})) \cap P(B,\mathfrak{A}) \neq \varnothing$, then $D^{\delta}(S_1(T_{\mathfrak{A}})) \cap P(B,\mathfrak{A}) \neq \varnothing$.*

PROOF. The proof of (i) and (ii) is trivial.

Since $D^{\nu}(S_1(T_{\mathfrak{A}}))$ is closed, and by (i), $P(B,\mathfrak{A})$ is closed, and $S_1(T_{\mathfrak{A}})$ is compact, $\bigcap_{\nu<\delta} (D^{\nu}(S_1(T_{\mathfrak{A}})) \cap P(B,\mathfrak{A})) = D^{\delta}(S_1(T_{\mathfrak{A}})) \cap P(B,\mathfrak{A}) \neq \varnothing$, and (iii) is proved.

LEMMA 7.8. *Let $\mathfrak{A}$ be ω-saturated, $a \in A$, and $K \in \mathscr{K}^a_{\mathfrak{A}}$;* let *$\mathfrak{K}$ be the submodel of $\mathfrak{A}$ with universe K. Then:*

(i) *The function $g(P(b,\mathfrak{K})) = P(b,\mathfrak{A})$, $b \in K$, from $S_1(T_{\mathfrak{K}})$ to $S_1(T_{\mathfrak{A}})$ is a homeomorphism.*

(ii) *If $b_1, b_2 \in K$ and $P(b_1,\mathfrak{A}) = P(b_2,\mathfrak{A})$ then $P(\langle a,b_1\rangle,\mathfrak{A}) = P(\langle a,b_2\rangle,\mathfrak{A})$.*

(iii) *The function f from $P(K,\mathfrak{A})$ to $S_2(T_{\mathfrak{A}})$, defined as*

$$f(P(b,\mathfrak{A})) = P(\langle a,b\rangle,\mathfrak{A}),\ b \in B,$$

is a homeomorphism into $S_2(T_{\mathfrak{A}})$.

PROOF. (i) By Lemma 7.6, g is continuous; by Corollary 3.8, g is one-to-one, and since its domain is compact g is a homeomorphism.

(ii) If K consists of a single point there is nothing to prove; otherwise, $\mathfrak{K}$ is SA, hence by Corollary 3.8, $P(b_1,\mathfrak{K}) = P(b_2,\mathfrak{K})$. By Theorem 2.1(ii), $P(\langle a,b_1\rangle,\mathfrak{A}) = P(\langle a,b_2\rangle,\mathfrak{A})$.

(iii) By Lemma 7.7(ii) the domain of f^{-1} is closed, clearly f^{-1} is continuous, and by (ii) f^{-1} is one-to-one, hence by Lemma 7.2, f^{-1} is a homeomorphism.

We omit the trivial proofs of the following lemmas.

LEMMA 7.9. *Let* $C = \{Q \mid Q \in S_n(T)$ *and* $Q \ni \bigwedge_{i=0}^{n-2}(v_i \leqq v_{i+1})\}$; *let* $f: C \to (S_2(T))^{n-1}$ *be defined as follows*: $f(Q) = \langle P_1,\cdots,P_{n-1}\rangle$ *where* $P_i = \{\psi \mid \psi \in Q$ *and* v_{i-1} *and* v_i *are the only free variables of* $\psi\}$, *then* f *is a homeomorphism into* $(S_2(T))^{n-1}$.

LEMMA 7.10. *If* $\mathfrak{A}$ *is* ω*-saturated*, $P \in S_1(T_{\mathfrak{A}})$, *and* $a \in A$, *then* $\{K \mid K \in \mathscr{K}^a_{\mathfrak{A}}$ *and* $P(K,\mathfrak{A}) \ni P\}$ *is closed in* $\mathscr{K}^a_{\mathfrak{A}}$.

LEMMA 7.11. *Let* $\mathfrak{A}$ *be* ω*-saturated*, $a \in A$, $K_i \in \mathscr{K}^a_{\mathfrak{A}}$, $i \in \omega$, $K \in \mathscr{K}^a_{\mathfrak{A}}$, $\lim_i K_i = K$ *where the limit is taken in* $\mathscr{K}^a_{\mathfrak{A}}$, $Q_i \in P(K_i,\mathfrak{A})$, $i \in \omega$, *and* $\lim_i Q_i = Q$; *then* $Q \in P(K,\mathfrak{A})$.

LEMMA 7.12. *Let* $\mathfrak{A}$ *be* ω*-saturated*, $a \in A$, $K \in \mathscr{K}^a_{\mathfrak{A}}$, *and* $a < b < K$; *then*:

(i) $K \in \mathscr{K}^b_{\mathfrak{A}}$.

(ii) *If* $\mathfrak{B} \subset \mathfrak{A}$ *and* $B \supseteq \{a\} \cup K$ *then* $K \in \mathscr{K}^a_{\mathfrak{B}}$.

PROOF. (i) Let $\mathfrak{K}$ be the submodel of $\mathfrak{A}$ with universe K. Let $K = |\mathfrak{A}|_{\Phi,a}$ where $\Phi \in \mathscr{F}^a_{\mathfrak{A}}$. By Corollary 2.4, $K = |\mathfrak{A}|_{\Phi_1,b}$ where each $\phi \in \Phi_1$ defines over b a convex set. By Lemma 4.9, either K contains a single element or $\mathfrak{K}$ is SA. Hence, by Lemma 4.10, $K \in \mathscr{K}^b_{\mathfrak{A}}$.

(ii) By Corollary 2.3, $K = |\mathfrak{B}|_{\Phi_2\,a}$ where each $\phi \in \Phi_2$ defines over a a convex set. Again, by Lemmas 4.9 and 4.10, $K \in \mathscr{K}^a_{\mathfrak{B}}$.

LEMMA 7.13. *Let* f *be an automorphism of* $\mathfrak{A}$, $a \in A$, *and* $f(a) > a$; *then there is an automorphism of* $\mathfrak{A}, g$ *such that* $g(a) = f(a)$, *and for every* $x \in A$, $g(x) \geqq x$.

PROOF. Define $C = \{x \mid$ for every $y \in \mathrm{conv}(\{a,x\})$ $f(y) \geqq y\}$; then clearly C is convex. It is easily seen that $f(C) = C$. Let

$$g(x) = \begin{cases} f(x) & x \in C, \\ x & x \notin C; \end{cases}$$

then clearly, g is the desired automorphism.

LEMMA 7.14. *Let $\mathfrak{A}$ be ω-saturated and ω-homogeneous, $a \in A$. $K_1, K_2 \in \mathscr{K}^a$, $a \leqq K_1 < K_2$, and $P \in P(K_1, \mathfrak{A}) \cap P(K_2, \mathfrak{A})$; then*

(i) $P(K_1, \mathfrak{A}) \subseteq P(K_2, \mathfrak{A})$.

(ii) *If there are K_3, $K_4 \in \mathscr{K}^a$ such that $P \in P(K_3, \mathfrak{A}) \cap P(K_4, \mathfrak{A})$ and $K_1 < K_3 < K_4 < K_2$ and $\{K \mid K \in \mathscr{K}^a$ and $K_1 \leqq K \leqq K_3$ and $P(K, \mathfrak{A}) \ni P\}$ is infinite, then $P(K_1, \mathfrak{A}) \subsetneqq P(K_2, \mathfrak{A})$.*

PROOF. (i) Since $P(K_1, \mathfrak{A}) \cap P(K_2, \mathfrak{A}) \neq \emptyset$ and $\mathfrak{A}$ is ω-homogeneous there is an automorphism f of $\mathfrak{A}$ which carries some element of K_1 to an element of K_2. By Lemma 7.13 it can be assumed that $f(a) \geqq a$. Obviously $f(K_1) \in \mathscr{K}^{f(a)}$; if $f(a) < K_2$ then by Lemma 7.12 (i) also $K_2 \in \mathscr{K}^{f(a)}$. Since $K_2 \cap f(K_1) \neq \emptyset$, necessarily $K_2 = f(K_1)$. If $f(a) \in K_2$ then for no $b \in K_1$, $f(b) < K_2$. Suppose there is some $b \in K_1$ such that $f(b) > K_2$. Let D be a convex set definable over a which contains K_2 and does not contain $f(b)$.

Let $\mathfrak{K}_1'$ be the submodel of $\mathfrak{A}$ with universe $f(K_1)$; then $D \cap f(K_1)$ is definable and convex in $\mathfrak{K}_1'$, and $\emptyset \neq D \cap f(K_1) \neq f(K_1)$. This contradicts the fact that $\mathfrak{K}_1'$ is SA; thus, it is impossible that $f(b) > K_2$ for some $b \in K_1$. In any case $f(K_1) \subseteq K_2$ and so $P(K_1, \mathfrak{A}) \subseteq P(K_2, \mathfrak{A})$.

(ii) Let $\mathscr{K} = \{K \in \mathscr{K}^a$ and $K_1 \leqq K \leqq K_2$ and $P(K, \mathfrak{A}) \ni P\}$. Suppose (ii) does not hold; then by (i) if $K \in \mathscr{K}$ then $P(K, \mathfrak{A}) = P(K_2, \mathfrak{A})$. Let $K \in \mathscr{K}$ and $K \neq K_1$; we define $A(K) = \{a \mid a < K$ and $K' < a$ for every $K' \in \mathscr{K}$ such that $K' < K\}$.

Since $\mathscr{K}$ is closed in $\mathscr{K}^a$ and $\mathscr{K}^a$ is complete,

$$\operatorname{conv}(K_1 \cup K_2) = (\bigcup \mathscr{K}) \cup \bigcup \{A(K) \mid K_1 \neq K \in \mathscr{K}\}.$$

If $P(A(K), \mathfrak{A}) \cap P(K_1, \mathfrak{A}) \neq \emptyset$ then there is some $K' \subseteq A(K)$ such that $P(K', \mathfrak{A}) \supseteq P(K_1, \mathfrak{A})$, thus $K' \in \mathscr{K}$; this contradicts the definition of $A(K)$. Hence $P(A(K), \mathfrak{A}) \cap P(K_1, \mathfrak{A}) = \emptyset$. Let $K \in \mathscr{K}$ and $K \neq K_1$; we show that $P(A(K), \mathfrak{A}) = P(A(K_2), \mathfrak{A})$. Let f be an automorphism of $\mathfrak{A}$ such that $f(K) \cap K_2 \neq \emptyset$. $f(A(K)) \cap K' = \emptyset$ for every $K' \in \mathscr{K}$, hence $f(A(K)) \subseteq A(K_2)$. Similarly $f^{-1}(A(K_2)) \subseteq A(K)$. Thus $f(A(K)) = A(K_2)$ whence $P(A(K), \mathfrak{A}) = P(A(K_2), \mathfrak{A})$.

Let $b_3 \in K_3$, $b_4 \in K_4$, and $P(b_3, \mathfrak{A}) = P(b_4, \mathfrak{A})$; let $\mathfrak{B}$ be the submodel of $\mathfrak{A}$ with universe $\{c \mid K_1 < c \in A\}$. Then by Corollary 3.7 and Theorem 2.1 (ii) $P(\langle a, b_3 \rangle, \mathfrak{A}) = P(\langle a, b_4 \rangle, \mathfrak{A})$. This contradicts the fact that b_3 and b_4 belong to distinct kernels over a. Hence $P(K_1, \mathfrak{A}) \subsetneqq P(K_2, \mathfrak{A})$. Q.E.D.

Lemma 7.15. *If $\mathfrak{A}$ is ω-homogeneous, $a \in A$, $K_1, K_2 \in \mathscr{K}^a$, $a \leqq K_1 < K_2$ and $P(K_1, \mathfrak{A}) \subsetneqq P(K_2, \mathfrak{A})$, then there exists an automorphism f of $\mathfrak{A}$ such that $f(\mathrm{conv}(\{a\} \cup K_1)) \subseteq K_2$.*

PROOF. Let f be an automorphism of $\mathfrak{A}$ such that $f(K_1) \cap K_2 \neq \varnothing$. Then, since $P(K_1, \mathfrak{A}) \subsetneqq P(K_2, \mathfrak{A})$, $f(K_1) \subsetneqq K_2$. If $f(a) \notin K_2$ then by Lemma 7.12(i), $K_2 \in \mathscr{K}^{f(a)}$, obviously, $f(K_1) \in \mathscr{K}^{f(a)}$; but since K_2 and $f(K_1)$ intersect and are not equal this situation is impossible. So $f(a) \in K_2$ and the lemma is proved.

We omit the proof of the following lemma.

LEMMA 7.16. *If $\mathfrak{A}$ is ω-saturated and ω-homogeneous, $a \in A, K_1, K_2 \in \mathscr{K}^a$, $a \leqq K_1 < K_2$, $P(K_1, \mathfrak{A}) \subseteq P(K_2, \mathfrak{A})$, and $R(K_1, \mathscr{K}^a) \neq R(K_2, \mathscr{K}^a)$, then there is an automorphism f of $\mathfrak{A}$ such that $f(\mathrm{conv}(\{a\} \cup K_1)) \subseteq K_2$.*

LEMMA 7.17. *Let $\mathfrak{A}$ be ω-saturated and ω-homogeneous, $S_1(T_{\mathfrak{A}})$ be atomic, $a \in A$, $P(a, \mathfrak{A})$ be isolated in $S_1(T_{\mathfrak{A}})$, $\mathfrak{C} \Subset \mathfrak{A}$, $K_i \in \mathscr{K}^a$, $i = 0, \cdots, 4$, $C < K_0 < K_1 < K_2 < K_3 < K_4$, $\min C = a$. $P(K_0, \mathfrak{A}) \subsetneqq P(K_1, \mathfrak{A}) \subseteq P(K_2, \mathfrak{A})$ and $P(K_3, \mathfrak{A}) \subsetneqq P(K_4, \mathfrak{A})$; then there is Q isolated in $S_1(T_{\mathfrak{A}})$ such that $Q \in P(K_4, \mathfrak{A})$ and $Q \notin P(C, \mathfrak{A})$.*

PROOF. We first show that there is a formula $\phi(v_0)$ and $c \in A$ such that $C < c \ngtr K_2$ and $\mathfrak{A} \models \phi[c]$ and for no $x \in C$, $\mathfrak{A} \models \phi[x]$. If not, then every n-type of a single element which is realized in $\mathrm{conv}(\{a\} \cup K_2)$ is realized in C. By Lemma 7.15 every n-type which is realized in C is realized in K_1. Hence every n-type which is realized in $\mathrm{conv}(\{a\} \cup K_2\rangle$ is realized in K_1. Let $b_i \in K_i$, $i = 1,2$, and $P(b_1, \mathfrak{A}) = P(b_2, \mathfrak{A})$; then since K_1 is SA and by Corollary 3.6 and Theorem 2.1(ii), $P(\langle a, b_1\rangle, \mathfrak{A}) = P(\langle a, b_2\rangle, \mathfrak{A})$. This is obviously a contradiction, hence there must be $c \in \mathrm{conv}(\{a\} \cup K_2)$ and $\phi(v_0)$ as required. Let $\phi(v_0)$ generate $P(a, \mathfrak{A})$ and $a \leqq |\mathfrak{A}|_{\psi,a} < K_3$, $c \in |\mathfrak{A}|_{\psi,a} \Subset \mathfrak{A}$. We define $\chi(v_0) \equiv \phi(v_0) \wedge \exists v_1(\alpha(v_1) \wedge \psi(v_0, v_1))$. There is some Q isolated in $S_1(T_{\mathfrak{A}})$ such that $Q \ni \chi$. Since α generates a type in $S_1(T_{\mathfrak{A}})$, Q must be realized in $|\mathfrak{A}|_{\psi,a}$ and, by Lemma 7.15, Q is realized in K_4. Since $Q \ni \phi$, $Q \notin P(C, \mathfrak{A})$ thus Q is as required and the lemma is proved.

LEMMA 7.18. (i) *If $\langle \mathscr{K}, < \rangle$ has a separating sequence and $\mathscr{K}_1 \subseteq \mathscr{K}$ then $\mathscr{K}_1$ has a separating sequence.*

(ii) *If $\langle \mathscr{K}, < \rangle$ is an ordered set of cardinality 2^ω and $\mathscr{K}$ has a separating sequence, then there is $\mathscr{K}_0 \subseteq \mathscr{K}$ with the following properties: $\mathscr{K}_0$ is of order type η, and for every $K_0 \in \mathscr{K}_0$ there are $\mathscr{K}'', \mathscr{K}' \subseteq \mathscr{K}$ both of order type η such that*

$$\{K \mid K_0 > K \in \mathscr{K}_0\} < \mathscr{K}' < K_0 < \mathscr{K}'' < \{K \mid K_0 < K \in \mathscr{K}_0\}.$$

PROOF. (i) is trivial.

To prove (ii) we first show that if $\langle A, < \rangle$ has a separating sequence and $\| A \| = 2^\omega$ then there is $a \in A$ such that $\| \, | \mathfrak{A}, a) \| = \| (a, \mathfrak{A} | \, \| = 2^\omega$. Let $\{\langle L_i, R_i \rangle\}_{i \in \omega}$ be a separating sequence for $\mathfrak{A}$. Let $L = \{a \mid a \in A \text{ and } \| \, | \mathfrak{A}, a) \| < 2^\omega\}$ and $R = \{a \mid a \in A \text{ and } \| (a, \mathfrak{A} | \, \| < 2^\omega\}$. Let $J = \{L_i \mid L_i \subsetneqq L\}$. If $a \in L$ and there is some $b > a$ in L, then $a \in L_i$ for some L_i in J. So $\| L - \bigcup J \| \leqq 1$. Since $\mathrm{cf}(2^\omega) > \omega$, $\| J \| \leqq \omega$, and each element of J is of cardinality less than 2^ω $\| \bigcup J \| < 2^\omega$, thus $\| L \| < 2^\omega$. Similarly $\| R \| < 2^\omega$, and hence $A - L \cup R \neq \emptyset$. It follows easily from what we proved that there is $\mathscr{K}' \subseteq \mathscr{K}$ of order type η; and it is easy to select $\mathscr{K}_0 \subseteq \mathscr{K}'$ with the desired properties.

LEMMA 7.19. *If $\langle X, < \rangle$ is complete and Y is a closed subset of X then the order topology on Y coincides with the relative topology induced on Y by X.*

PROOF. Notice that the identity function is continuous from Y with the relative topology, to Y with the order topology. Since both topologies are Hausdorff and compact, the identity mapping must be a homeomorphism, hence the topologies must coincide.

DEFINITION. Let $\langle A, < \rangle$ be an ordered set. $B \subseteq A$ is called n-disjointed in A if for every $b_0 \in B$ there are $C_1, C_2 \subseteq A$ such that

$$\{b \mid b_0 > b \in B\} < C_1 < b_0 < C_2 < \{b \mid b_0 < b \in B\}$$

and $\| C_1 \| = \| C_2 \| = n$.

LEMMA 7.20. *Let $\langle X, < \rangle$ be countable and complete, $a \in X$, $1 \leqq R(a, X)$; then there is $B \subseteq X$ with the following properties.*

(i) $a \notin B$.

(ii) *$B \cup \{a\}$ is complete.*

(iii) *B is n-disjointed.*

(iv) $a \in \mathrm{cl}(B, X)$.

(v) *Let B_0 be the topological space with $B \cup \{a\}$ as the underlying set and with the order topology; then $R(a, B_0) = R(a, X)$.*

PROOF. We prove the lemma by induction on $R(a, X)$. For $R(a, X) = 1$ the proof is trivial. Suppose $R(a, X) = \nu > 1$. Let $\{y_i\}_{i \in \omega}$ be a strictly monotone sequence such that: $\lim_i y_i = a$,

$$\text{if } i > j \text{ then } R(y_i, X) \geqq R(y_j, X),$$

$$\text{and } \nu = \min\{\xi \mid \xi > R(y_i, X) \text{ for every } i \in \omega\}.$$

Without loss of generality, $y_i < a$ for every i. By selecting a subsequence we can assume that either for every i, $R(y_i, |\, y_i]) = R(y_i, X)$ or for every i, $R(y_i, [y_i\,|) = R(y_i, X)$. We prove the lemma in the first case. Since $R(y_i, |\, y_i]) < \nu$, by the induction hypothesis, there is $A_i \subseteq |\, y_i]$ with properties (i)–(v). If $i > 0$ let z_i be with the following properties

(i) $z_i \in A_i$.

(ii) There is $X' \subseteq X$ such that $\| X' \| = n$ and $y_{i-1} < X' < z_i$.

(iii) z_i has a successor in A_i.

(iv) There is no $z \in A_i$ such that $z_i < z$ and $R(z, A_i \cup \{y_i\}) > R(y_i, A_i \cup \{y_i\})$, where the order topology is taken on $A_i \cup \{y_i\}$.

Let $B = \bigcup_{i>0}(A_i \cap [z_i|)$; it is easy to show that B has the desired properties. In the case where for every i, $R(y_i, [y_i\,|) = R(y_i, X)$, the construction of B is similar.

THEOREM 7.21. *If* $\| S_1(T) \| \leqq \omega$ *then* $\| S_2(T) \| \leqq \omega$.

PROOF. Suppose the theorem is not true and T' is such that $\| S_1(T') \| \leqq \omega$ and $\| S_2(T') \| > \omega$. We shall prove the following statements.

(i) Let $\mathfrak{B}$ be an ω-saturated model of T'; then there is $b \in B$ and $\mathscr{K}' \subseteq \mathscr{K}^b$ such that $\| \mathscr{K}' \| = 2^\omega$ and $\bigcap \{P(K, \mathfrak{B}) \mid K \in \mathscr{K}'\} \neq \varnothing$. Using (i), we obtain (ii).

(ii) There is T for which $\| S_1(T) \| \leqq \omega$ and $\| S_2(T) \| = 2^\omega$, and there is an ω-saturated model of T, $\mathfrak{A}$, $a \in A$, and $\mathscr{K} \subseteq \mathscr{K}^a$ such that $P(a, \mathfrak{A})$ is isolated in $S_1(T)$, $\| \mathscr{K} \| = 2^\omega$, and $\bigcap \{P(K, \mathfrak{A}) \mid K \in \mathscr{K}\} \neq \varnothing$.

(iii). Let T, $\mathfrak{A}$, a, $\mathscr{K}$ be as in (ii), and let $\mathscr{K}_0$ be a subset of $\mathscr{K}$ with properties as in Lemma 7.18 (ii). We shall prove that for every $K \in \mathscr{K}_0$ there is Q_K isolated in $S_1(T)$ such that $Q_K \in P(K, \mathfrak{A})$ and

$$Q_K \notin P(\mathrm{conv}(\{a\} \cup \bigcup \{K' \mid K' < K \text{ and } K' \in \mathscr{K}_0\}), \mathfrak{A}).$$

(iv) Using (iii) we shall construct 2^ω types of $S_1(T)$.

Since (iv) contradicts (ii) the existence of T' as above is impossible, and the theorem thus will be proved.

(i) By Theorem 7.1 (ii), $\| S_2(T') \| = 2^\omega$, hence there must be some $P \in S_1(T')$ such that $\| \{Q \mid Q \in S_2(T') \text{ and } Q \supseteq P\} \| = 2^\omega$. Let $\mathfrak{B}$ be an ω-saturated, ω-homogeneous model of T'. We choose $b \in B$ such that $P(b, \mathfrak{B}) = P$.

Since $\| S_1(T') \| \leqq \omega$, by Lemma 7.8(ii), $\| \mathscr{K}^b \| = 2^\omega$. For every $P \in S_1(T')$, let $\mathscr{K}_P = \{K \mid K \in \mathscr{K}^b \text{ and } P(K, \mathfrak{B}) \ni P\}$. Again since $\| S_1(T') \| \leqq \omega$ there must be some $\mathscr{K}_P$ of power 2^ω. Thus (i) is proved.

(ii) $\mathscr{K}^b$ has a separating sequence, thus by Lemma 7.18(i), $\mathscr{K}_P$ has a separating sequence and by Lemma 7.18(ii) there is a subset of $\mathscr{K}_P$ which is of order type η. Denote this subset by $\mathscr{K}'$. Without loss of generality, $b < K$ for each $K \in \mathscr{K}'$. Let $\mathscr{K}' = \{K'_r\}_{r \in Q}$ where $r \to K'_r$ is an order isomorphism. Let α be a real number and $K_\alpha = \sup\{K'_r \mid r < \alpha\}$ where the supremum is taken in $\mathscr{K}^b$. By Lemmas 7.10, 7.14, and 7.15, $P(K_\alpha, \mathfrak{B}) \supseteq P(\{c \mid b \leqq c < K_\alpha\}, \mathfrak{B})$. There must be some α for which $P(K_\alpha, \mathfrak{B}) = P(\{c \mid b \leqq c < K_\alpha\}, \mathfrak{B})$ since otherwise $\| S_1(T') \| = 2^\omega$. Let $\mathfrak{K}_\alpha$ be the submodel of $\mathfrak{B}$ whose universe is K_α. By Corollary 3.8 and since $\mathfrak{K}_\alpha$ is ω-saturated, $\| S_1(T_{\mathfrak{K}_\alpha}) \| \leqq \omega$. Denote $T = T_{\mathfrak{K}_\alpha}$; then there is $a' \in K_\alpha$ such that $P(a', \mathfrak{K}_\alpha)$ is isolated in $S_1(T)$. By the choice of K_α there is some a such that $b \leqq a < K_\alpha$ and $P(a, \mathfrak{B}) = P(a', \mathfrak{B})$. Since $\mathfrak{B}$ is ω-homogeneous there is an automorphism f of $\mathfrak{B}$ such that $f(a) = a'$. We show that if $K \in \mathscr{K}^b$ and $a \leqq K < K_\alpha$ then $f(K) \subseteq K_\alpha$. First, notice that for every $K \in \mathscr{K}^b$: if $a \leqq K < K_\alpha$, then $P(K, \mathfrak{B}) \subsetneqq P(K_\alpha, \mathfrak{B})$, for otherwise $P(K'_r, \mathfrak{B}) = P(K_\alpha, \mathfrak{B})$ for some $r < \alpha$ which, by Lemma 7.14, is impossible. Let $Q \in P(K_\alpha, \mathfrak{B})$. Since $\mathfrak{K}_\alpha$ is ω-saturated and SA, for every $c \in K_\alpha$ there is $c' > c$ which realizes Q. Set $K' = \sup\{K \mid K \in \mathscr{K}^b \text{ and } f(K) \cap K_\alpha \neq \varnothing\}$ where the supremum is taken in $\mathscr{K}^b$. If K' is the maximum of the above set then $P(K', \mathfrak{B}) \supseteq P(K_\alpha, \mathfrak{B})$. If K' is not the maximum then by Lemma 7.10 again $P(K', \mathfrak{B}) \supseteq P(K, \mathfrak{B})$, but this is possible only if $K' \geqq K_\alpha$; thus if $a \leqq K < K_\alpha$ then $f(K) \subseteq K_\alpha$.

Let $\mathfrak{A}$ be the submodel of $\mathfrak{B}$ whose universe is $f^{-1}(K_\alpha)$. By Lemma 7.12(ii) if $K \in \mathscr{K}^a_{\mathfrak{B}}$ and $K \subseteq A$ then $K \in \mathscr{K}^a_{\mathfrak{A}}$. By Lemma 7.12(i) if $K'_r > a$ then $K'_r \in \mathscr{K}^a_{\mathfrak{B}}$. Combining the last facts we obtain that if $r < \alpha$ and $K'_r > a$ then $K'_r \in \mathscr{K}^a_{\mathfrak{A}}$. Since $\mathfrak{A}$ is ω-saturated, $\mathscr{K}^a_{\mathfrak{A}}$ is complete, hence $\| \mathscr{K}^a_{\mathfrak{A}} \| = 2^\omega$. Since $P(a', \mathfrak{K}_\alpha)$ is isolated in $S_1(T)$ the same is true for $P(a, \mathfrak{A})$. It is now easy to select $\mathscr{K} \subseteq \mathscr{K}^a_{\mathfrak{A}}$ as required in (ii). T, $\mathfrak{A}$, a, $\mathscr{K}$ have the properties mentioned in (ii).

(iii) Let $\mathscr{K}_0$ be a subset of $\mathscr{K}$ with properties as in Lemma 7.18(ii). Let $\{K_r\}_{r \in Q} = \mathscr{K}_0$ where $r \to K_r$ is an order isomorphism, and let $C_r = \{c \mid a \leqq c$ and $c < K_q$ for some $q < r\}$. For every $r \in \mathbb{Q}$ let $\mathscr{K}'_r \subseteq \mathscr{K}$ be of order type η, and if $K' \in \mathscr{K}'$ then $C_r < K' < K_r$. We select K^0, K^1, K^2, K^3 in $\mathscr{K}'_r$ such that $K^0 < K^1 < K^2 < K^3$. $\mathscr{K}$ was chosen so that $P(K_r, \mathfrak{A}) \supseteq P(K^3, \mathfrak{A}) \supseteq P(K^2, \mathfrak{A}) \supseteq P(K^1, \mathfrak{A}) \supseteq P(K^0, \mathfrak{A})$; by Lemma 7.14 the inclusions are all proper, and by Lemma

7.17 there is Q_r isolated in $S_1(T)$ such that $Q_r \in P(K_r, \mathfrak{A}) - P(C_r, \mathfrak{A})$. Hence (iii) is proved.

(iv) Let $c \in A$ and $P(c, \mathfrak{A}) = Q_r$. We show that c has the following property. If $q_1, q_2 < r$ and $c_1 > c$ and $P(c_1, \mathfrak{A}) = Q_{q_1}$ then there is c_2 such that $c < c_2 < c_1$ and $P(c_2, \mathfrak{A}) = Q_{q_2}$. If $q = \max(q_1, q_2)$ then $\mathfrak{K}_q$ is ω-saturated, SA, and $P(K_q, \mathfrak{A}) \ni Q_{q_1}, Q_{q_2}$ and $P(K_q, \mathfrak{A}) \not\ni Q_r$. Let $c_1 > c$ and $P(c_1, \mathfrak{A}) = Q_{q_1}$ Since $\mathfrak{A}$ is ω-homogeneous and $P(K_q, \mathfrak{A}) \ni P(c_1, \mathfrak{A})$ there is an automorphism f of $\mathfrak{A}$ such that $f(K_q) \ni c_1$, but $f(K_q) \not\ni c$ since $P(K_q, \mathfrak{A}) \not\ni P(c, \mathfrak{A})$. By Corollary 3.3 there is $c_2 < c_1$, $c_2 \in f(K_q)$ such that $P(c_2, \mathfrak{A}) = Q_{q_2}$. Certainly $a < c_2$. On the other hand, if $r < q_1 < q_2$, obviously there is $c_1 > c$ such that $P(c_1, \mathfrak{A}) = Q_{q_1}$ and $P((c, c_1), \mathfrak{A}) \not\ni Q_{q_2}$.

Let $\phi_{q_1q_2}(v_0)$ be the formula expressing the following property of v_0. If $c_1 > v_0$ and $P(c_1, \mathfrak{A}) = Q_{q_1}$ then there is c_2 such that $v_0 < c_2 < c_1$ and $P(c_2, \mathfrak{A}) = Q_{q_2}$ and if $c_1 > v_0$ and $P(c_1, \mathfrak{A}) = Q_{q_2}$ then there is c_2 such that $v_0 < c_2 < c_1$ and $P(c_2, \mathfrak{A}) = Q_q$. Such a formula exists since Q_q is isolated in $S_1(T)$. For each irrational α let

$$P_\alpha = \{\phi_{q_1q_2}(v_0) \mid q_1, q_2 < \alpha\} \cup \{\sim\phi_{q_1q_2}(v_0) \mid q_1, q_2 > \alpha\}.$$

Since $Q_r \ni \phi_{q_1q_2}$ for every $q_1, q_2 > r$ and $Q_r \not\ni \sim\phi_{q_1q_2}$ for every $q_1, q_2 > r$ then P_α is finitely satisfiable. Obviously, if $\alpha \neq \beta$ then $P_\alpha \cup P_\beta$ is contradictory. Since each P_α can be extended to a type in $S_1(T)$, $\| S_1(T) \| = 2^\omega$. This is a contradiction to the fact that $\| S_1(T) \| \leqq \omega$ and the theorem is proved.

COROLLARY 7.22. *If* $\| S_1(T) \| \leqq \omega$, *then for every* n, $\| S_n(T) \| \leqq \omega$.

The proof is trivial by Lemma 7.9.

REMARK. Another question could be asked in the same connection. If $F_1(T)$ is atomic, does the same fact hold for $F_n(T)$? The answer to this question is negative: there is T for which $F_1(T)$ is atomic and $F_2(T)$ is atomless.

DEFINITION. Let $\mathfrak{A}$ be a model, $a \in A$ and $V \subseteq A$. V is a halfneighbourhood of a if there exists a convex neighbourhood W of a such that either $V = W \cap [a|$ or $V = W \cap |a]$.

DEFINITION. Let $\mathfrak{A}$ be a model, $B \subseteq A$. $P \in S_1(T_{\mathfrak{A}})$ is B, $\mathfrak{A}$ isolated, if P is realized in B, and there exists $\phi(v_0)$ such that for every $b \in B$, $\mathfrak{A} \models \phi[b]$ iff $\mathfrak{A} \models P[b]$. If ν is an ordinal and $\nu < \omega_1$ we define $d(\nu) = \bigcup \{\xi \mid D^\xi(\omega_1) \cap (\nu + 1) \neq \varnothing\}$. We define a sequence $\{a_\nu\}_{\nu<\omega_1}$: $a_0 = 1$; $a_{\nu+1} = a_\nu + 1 + \nu + 1 + d(\nu) + 1$; if ν is a limit ordinal then $a_\nu = \bigcup_{\xi<\nu} a_\xi$.

THEOREM 7.23. *Let $\mathfrak{A}$ be a countable, saturated model, $a \in A$, and $L_0 \in D^{a_\nu}(\mathscr{K}^a)$, then $P(L_0, \mathfrak{A}) \cap D^\nu(S_1(T_\mathfrak{A})) \neq \varnothing$.*

PROOF. If $\mathscr{K} \subseteq \mathscr{K}^a$ let $\mathscr{K}^{Or}$ denote $(\mathscr{K}, \tau)$ where τ is the order topology on $\mathscr{K}$. $\mathscr{K}$ will denote $(\mathscr{K}, \tau')$ where τ' is the relative topology induced on $\mathscr{K}$ by $\mathscr{K}^a$. We prove the following claim by induction on ν: for every countable, saturated model $\mathfrak{A}$, for every $a \in A$, and for every $L_0 \in D^{a_\nu}(\mathscr{K}^a)$: $P(L_0, \mathfrak{A}) \cap D^\nu(S_1(T_\mathfrak{A})) \neq \varnothing$. The claim is trivially true for $\nu = 0$. By Lemma 7.7 and the induction hypothesis it is true for ν a limit ordinal.

Assume the induction hypothesis for ν. We shall prove the following statement.

Statement I. If $K_0 \in D^{a_\nu+1+\nu+1}(\mathscr{K}^a)$ and there exists a halfneighbourhood V_1 of K_0 such that

(i) $R(K_0, V_1) = R(K_0, \mathscr{K}^a)$ and

(ii) if $B_1 = \bigcup V_1 - K_0$, $C \Subset B_1$, C is definable over a and $\phi(v_0)$ is realized in C, then there exists $P \in S_1(T_\mathfrak{A})$ such that $\phi \in P$, P is isolated and P is realized in C.

Then $P(K_0, \mathfrak{A}) \cap D^{\nu+1}(S_1(T_\mathfrak{A})) \neq \varnothing$.

Proof of Statement I. Without loss of generality, $a < K_0$. Assume, by way of contradiction, $D^{\nu+1}(S_1(T_\mathfrak{A})) \cap P(K_0, \mathfrak{A}) = \varnothing$. We show that there exists a halfneighbourhood of K_0, $V \subseteq V_1$ with the following properties:

(i) V is a closed subset of $\mathscr{K}^a$.

(ii) $\bigcup V > a$.

(iii) Let $B = \bigcup V - K_0$; then for every $C \Subset B$ and for every $\phi(v_0)$ if C is definable over a and ϕ is realized in C, then there exists $P \in P(C, \mathfrak{A})$ such that P is isolated and $P \ni \phi$.

(iv) For every ξ, $D^\xi(V) = D^\xi(\mathscr{K}^a) \cap V$.

(v) $P(\bigcup V, \mathfrak{A}) \cap D^\nu(S_1(T_\mathfrak{A})) = \mathscr{P}$ is finite.

Since $\mathfrak{A}$ is ω-saturated and $P(K_0, \mathfrak{A}) \cap D^{\nu+1}(S_1(T_\mathfrak{A})) = \varnothing$, there is a neighbourhood W of K_0 such that $P(\bigcup W, \mathfrak{A}) \cap D^\nu(S_1(T_\mathfrak{A}))$ is finite. Since $\mathscr{K}^a$ is totally disconnected there is $V \subseteq W$, where V is a halfneighbourhood of K_0 such that V is closed and open in V_1, then V satisfies (i)–(v). Thus we have proved the existence of V.

For every $Q \in \mathscr{P}$ we define $\mathscr{K}_Q$: $\mathscr{K}_Q = \{K \mid K \in D^{a_\nu}(V),\ Q \in P(K, \mathfrak{A})\}$, then for every Q, $\mathscr{K}_Q$ is closed in V. By the induction hypothesis and by (iv), $D^{a_\nu}(V) = \bigcup_{Q \in \mathscr{P}} \mathscr{K}_Q$. By (iv) and Lemma 7.4, $K_0 \in D^{1+\nu+1}(D^{a_\nu}(V))$; by Lemma 7.5 there exists $Q' \in \mathscr{P}$ such that $R(K_0, \mathscr{K}_{Q'}) = R(K_0, D^{a_\nu}(V))$. Let $\mathscr{K}' = \mathscr{K}_{Q'}$. We show that there exists $\underline{\mathscr{K}} \subseteq \mathscr{K}'$ with the following properties.

(i) $\underline{\mathscr{K}}$ is closed in $D^{a_v}(V)$.

(ii) $\mathrm{int}(\underline{\mathscr{K}}, D^{a_v}(V)) \supseteq \underline{\mathscr{K}} - \{K_0\}$.

(iii) $\underline{\mathscr{K}}$ is a neighbourhood of K_0 relative to $\mathscr{K}'$ and $\underline{\mathscr{K}} \in \mathscr{K}'$.

Suppose first that $K_0 \leqq V$. Let $K' \in \mathscr{K}'$ such that if $K_0 < K_i \leqq K'$ and $K_i \in \mathscr{K}'$ for $i = 1,2$, then $P(K_1, \mathfrak{A}) \cap D^v(S_1(T_{\mathfrak{A}})) = P(K_2, \mathfrak{A}) \cap D^v(S_1(T_{\mathfrak{A}}))$, and such that $[K_0, D^{a_v}(V), K']$ is closed and open in $D^{a_v}(V)$. Let $\underline{K} = \mathscr{K}' \cap [K_0, D^{a_v}(V), K']$; then $\underline{\mathscr{K}}$ has the desired properties: obviously $\underline{\mathscr{K}}$ has properties (i) and (iii); suppose $\tilde{K} \in \underline{\mathscr{K}} - \{K_0\}$ but $\tilde{K} \notin \mathrm{int}(\underline{\mathscr{K}}, D^{a_v}(V))$, we have a sequence $\{K_i\}_{i \in \omega}$ in $[K_0, \mathscr{K}^a, K'] \cap D^{a_v}(V) - \underline{\mathscr{K}}$ such that $\lim_i K_i = \tilde{K}$. $P(K_i, \mathfrak{A}) \cap \mathscr{P} \neq \emptyset$, so we may assume that there is some Q in $\mathscr{P}$ which belongs to $P(K_i, \mathfrak{A})$ for every i; hence $Q \in P(\tilde{K}, \mathfrak{A})$, but then $Q \in P(K, \mathfrak{A})$ for all K in $\underline{\mathscr{K}}$. Take $K_1 > K \in \underline{\mathscr{K}}$; then $P(K, \mathfrak{A}) \subseteq P(K_1, \mathfrak{A})$ and $K_1 \in \underline{\mathscr{K}}$ contradicting the choice of K_1. If $V \leqq K_0$ we define $\underline{\mathscr{K}}$ similarly.

Let $K \in \underline{\mathscr{K}}$. If $K \neq K_0$ then $R(K, \underline{\mathscr{K}}) = R(K, D^{a_v}(V))$ and by Lemma 7.4 and the choice of V, $R(K, \mathscr{K}^a) = a_v + R(K, \underline{\mathscr{K}})$. By (iii) and by the choice of $\mathscr{K}'$, $R(K_0, \underline{\mathscr{K}}) = R(K_0, \mathscr{K}') = R(K_0, D^{a_v}(V))$; again $a_v + R(K_0, \underline{\mathscr{K}}) = R(K_0, \mathscr{K}^a)$ so that for $K_1, K_2 \in \underline{\mathscr{K}}$, $R(K_1, \underline{\mathscr{K}}) = R(K_2, \underline{\mathscr{K}})$ iff $R(K_1, \mathscr{K}^a) = R(K_2, \mathscr{K}^a)$.

Let $K_i \in \underline{\mathscr{K}}$, $K_i \neq K_0$, $i = 1,2,3,4$, $K_1 < K_2 < K_3 < K_4$, and $R(K_1, \underline{\mathscr{K}}) \neq R(K_2, \underline{\mathscr{K}})$. We show that there exists a formula $\phi(v_0)$ which is satisfied by some element of $\mathrm{conv}(\{a\} \cup K_3)$ and by no element of $\mathrm{conv}(\{a\} \cup K_1)$. If not, then by Lemma 7.16 if $Q_1, \cdots, Q_r$ are n-types which are realized in $\mathrm{conv}(\{a\} \cup K_3)$ then $\langle Q_1, \cdots, Q_r \rangle$ is realized in K_2. Take b_2, b_3 in K_2, K_3 respectively such that $P(b_2, \mathfrak{A}) = P(b_3, \mathfrak{A})$; then by Corollary 3.6, $P(\langle a, b_1 \rangle, \mathfrak{A}) = P(\langle a, b_2 \rangle, \mathfrak{A})$ which is a contradiction. By the third property of V there exists an isolated P in $S_1(T_{\mathfrak{A}})$ and some c such that $K_1 < c < K_4$ and P is not realized in $\mathrm{conv}(\{a\} \cup K_1)$ and $\mathfrak{A} \models P[c]$.

Let $\overline{\mathscr{K}} = D^1(\underline{\mathscr{K}})$. Since $\overline{\mathscr{K}}$ is closed in $\mathscr{K}^a$, by Lemma 7.19 $\overline{\mathscr{K}} = \overline{\mathscr{K}}^{Or}$. So by Lemma 7.20 there exists $\mathscr{K} \subseteq \overline{\mathscr{K}}$ such that:

(i) $K_0 \notin \mathscr{K}$.

(ii) Let $\mathscr{K}_0 = \mathscr{K} \cup \{K_0\}$; then $\mathscr{K}_0$ is complete.

(iii) $\mathscr{K}$ is 4-disjointed in $\overline{\mathscr{K}}$.

(iv) $K_0 \in \mathrm{cl}(\mathscr{K}, \overline{\mathscr{K}}^{Or}) = \mathrm{cl}(\mathscr{K}, \overline{\mathscr{K}})$.

(v) $R(K_0, \mathscr{K}_0^{Or}) = R(K_0, \overline{\mathscr{K}})$.

Let $K \in \mathscr{K}$; we define a formula ϕ_K and a type P_K. Since $\mathscr{K}$ is 4-disjointed there are K_i in $\overline{\mathscr{K}}$, $1 \leqq i \leqq 4$, such that $\{K' \mid \mathscr{K}' \ni K' < K\} < K_1 < K_2 < K_3 < K_4 < K$. By Theorem 7.1 there is ξ such that $D^\xi(\underline{\mathscr{K}}) = \varnothing$ so there are infinitely many isolated kernels of $\underline{\mathscr{K}}$ between K_1 and K_3. We choose three of those: $K_1 < K^2 < K^3 < K^4 < K_3$. Since $R(K_1, \underline{\mathscr{K}}) \neq R(K^2, \underline{\mathscr{K}})$, there is P_K isolated in $S_1(T_{\mathfrak{A}})$ which is not realized in $\mathrm{conv}(\{a\} \cup K_1)$ but is realized in K_3. Let ϕ_K generate P_K. Let $K \in \mathscr{K}$ and K' be its successor in $\mathscr{K}$. We define

$$\psi_K(v_0) \equiv \forall x \exists y(((x > v_0 \wedge \phi_K(x)) \rightarrow (v_0 < y < x \wedge \phi_{K'}(y)))$$
$$\wedge ((v_0 < x \wedge \phi_{K'}(x)) \rightarrow (v_0 < y < x \wedge \phi_K(y)))).$$

For $K \in \mathscr{K}_0$ let $\Psi_K = \{\psi_L \mid \mathscr{K} \ni L < K$ and L has a successor in $\mathscr{K}\} \cup \{\sim\psi_L \mid \mathscr{K} \ni L \geqq K$ and L has a successor in $\mathscr{K}\}$. We claim:

(i) For every $K \in \mathscr{K}_0$ there exists $P \in S_1(T_{\mathfrak{A}})$ such that $P \supseteq \Psi_K$.

(ii) If $K_1 < K_2$ are kernels in $\mathscr{K}$ then there is $\psi_K(v_0)$ such that $\psi_K \in \Psi_{K_2}$ and $\sim\psi_K \in \Psi_{K_1}$.

(iii) If $K \neq K_0$ and $K \in D^\xi(\mathscr{K}_0^{0r})$ then there exists $Q_K \in D^\xi(S_1(T_{\mathfrak{A}}))$, $\Psi_K \subseteq Q_K$, and there exists c such that $K \ngtr c < \bigcup\{K' \mid K < K' \in \mathscr{K}\}$ and $\mathfrak{A} \models Q_K[c]$.

To prove (i), it suffices to show that Ψ_K is realized by some element of $\mathfrak{A}$. Let $K \in \mathscr{K}$, and $K_1 < K_2 < K_3 < K_4 < K$ be the kernels belonging to $\overline{\mathscr{K}} - \mathscr{K}$, such that $P_K \in P(K_3, \mathfrak{A})$ and $P_K \notin P(\mathrm{conv}(\{a\} \cup K_1), \mathfrak{A})$. Choose $c \in K$ such that $P(c, \mathfrak{A}) \notin P(\mathrm{conv}(\{a\} \cup K_3), \mathfrak{A})$; then $\mathfrak{A} \models \Psi_K[c]$, because all the P_L's for $L < K$ are realized in K_2 which is SA. Since the Ψ_K's are realized Ψ_{K_0} is finitely satisfiable in $\mathfrak{A}$, hence there is $P \in S_1(T_{\mathfrak{A}})$ containing Ψ_K, for every $K \in \mathscr{K}_0$. We omit the trivial proof of (ii).

To prove (iii) we define for every $K \in \mathscr{K}$, $K^>$; if K has a successor in $\mathscr{K}$ then $K^> = K$, otherwise,

$K^> = K \cup \{b \mid K < b$ and for every $K' \in \mathscr{K}$, if $K' > K$ then $b < K'\}$. $K^>$ is the intersection of convex sets definable over a. We prove by induction on ξ that, when $\xi < \omega_1$ and $K \in D^\xi(\mathscr{K}_0^{0r})$ and $K \neq K_0$, then there is $P \in D^\xi(S_1(T_{\mathfrak{A}}))$ such that $\Psi_K \subseteq P \in P(K^>, \mathfrak{A})$. There is nothing to prove when $\xi = 0$. If ξ is a limit ordinal $K \in D^\xi(\mathscr{K}_0^{0r})$ and $K \neq K_0$, then, by the induction hypothesis, for every $\eta < \xi$ there is $Q_\eta \in D^\xi(S_1(T_{\mathfrak{A}}))$ such that $Q_\eta \supseteq \Psi_K$ and $Q_\eta \in P(K^>, \mathfrak{A})$. Let Q be a limit point of $\{Q_\eta\}_{\eta<\xi}$; then it is easily seen that $Q \supseteq \Psi_K$, $Q \in P(K^>, \mathfrak{A})$, and $Q \in D^\xi(S_1(T_{\mathfrak{A}}))$.

Assume the induction hypothesis for ξ, and let $K \in D^{\xi+1}(K_0^{0r})$ and $K \neq K_0$. Let $\{L_i\}_{i \in \omega}$ be a strictly monotone sequence of kernels in $D^\xi(\mathscr{K}_0^{0r})$ such that

$\lim_i L_i = K$ where the limit is taken in $\mathscr{K}_0^{0r}$. By the induction hypothesis we have Q_i such that $Q_i \in P(L_i, \mathfrak{A}) \cap D^{\xi}(S_1(T_{\mathfrak{A}}))$ and $Q_i \supseteq \Psi_{L_i}$. Without loss of generality, $\lim_i Q_i = Q_i$. By (ii), $Q_i \neq Q_j$ for $i \neq j$ so that $Q \in D^{\xi+1}(S_1(T_{\mathfrak{A}}))$. Clearly $Q \supseteq \Psi_K$. To see that Q is realized in $K^{>}$ consider first the case when $L_i > K$, then let L be the infimum of the L_i's in $\mathscr{K}^a$; then $L \subseteq K^{>}$, and obviously Q is realized in L. If $L_i < K$ let L be the supremum of the L_i's in $\mathscr{K}^a$; then Q is realized in L. We remember that $\bigcap_i P(L_i, \mathfrak{A}) \cap P(K, \mathfrak{A}) \neq \varnothing$ so that by Lemma 7.11, $P(L, \mathfrak{A}) \cap P(K, \mathfrak{A}) \neq \varnothing$, so $P(L, \mathfrak{A}) \subseteq P(K, \mathfrak{A})$, thus $Q \in P(K, \mathfrak{A})$. We may now show that K_0 has a similar property; that is, since as we shall see, $K_0 \in D^{\nu+1}(\mathscr{K}_0^{0r})$, there is some $Q \in D^{\nu+1}(S_1(T_{\mathfrak{A}}))$ such that $Q \supseteq \Psi_{K_0}$ and Q is realized in K_0.

$K_0 \in D^{a_\nu+1+\nu+1}(\mathscr{K}^a)$, hence $K_0 \in D^{a_\nu+1+\nu+1}(V)$. By Lemma 7.4, $K_0 \in D^{1+\nu+1}(D^{a_\nu}(V))$. We chose $\mathscr{K}'$ so that $K_0 \in D^{1+\nu+1}(\mathscr{K}')$. Since $\underline{\mathscr{K}}$ is a neighbourhood of K_0 relative to $\mathscr{K}'$, $K_0 \in D^{1+\nu+1}(\underline{\mathscr{K}})$; again by Lemma 7.4, $K_0 \in D^{\nu+1}(\overline{\mathscr{K}})$ hence $K_0 \in D^{\nu+1}(\mathscr{K}_0^{0r})$. To show the existence of Q with the properties mentioned, we merely have to repeat the same arguments.

We deduced that $D^{\nu+1}(S_1(T_{\mathfrak{A}})) \cap P(K_0, \mathfrak{A}) \neq \varnothing$ in contradiction to our assumption; hence it must be the case that $D^{\nu+1}(S_1(T_{\mathfrak{A}})) \cap P(K_0, \mathfrak{A}) \neq \varnothing$.

Now that we have proved Statement I we proceed to the general case. Let $L_0 \in D^{a_\nu}(\mathscr{K}^a)$. We may assume that $a < L_0$. We define $V_1 = \{K \mid K \in \mathscr{K}^a$ and $a \leqq K \leqq L_0\}$ and $V_2 = \{K \mid K \in \mathscr{K}^a$ and $L_0 \leqq K\}$; then by Lemma 7.5 either $R(L_0, V_1) = R(L_0, \mathscr{K}^a)$ or $R(L_0, V_2) = R(L_0, \mathscr{K}^a)$. We consider the first case. Assume by way of contradiction that $P(L_0, \mathfrak{A}) \cap D^{\nu+1}(S_1(T_{\mathfrak{A}})) = \varnothing$. Let $B = \bigcup V_1 - L_0$; we may assume that $P(a, \mathfrak{A})$ is $B, \mathfrak{A}$ isolated for otherwise we replace a by $b \in B$ such that $P(b, \mathfrak{A})$ is $B, \mathfrak{A}$ isolated, and by Lemma 7.12(i), $L_0 \in D^{a_\nu+1}(\mathscr{K}^b)$. Let $V \subset V_1$ be a halfneighbourhood of L_0 with the following properties:

(i) V is closed in $\mathscr{K}^a$.

(ii) For all ξ, $D^{\xi}(V) = D^{\xi}(\mathscr{K}^a) \cap V$.

(iii) $D^{\nu}(S_1(T_{\mathfrak{A}})) \cap P(\bigcup V, \mathfrak{A}) = \mathscr{P}$ is finite.

(iv) $D^{\nu+1}(S_1(T_{\mathfrak{A}})) \cap P(\bigcup V, \mathfrak{A}) = \varnothing$.

We now choose $\mathscr{K}'$ and $\underline{\mathscr{K}}$ as before; then $L_0 \in D^{1+\nu+1+d(\nu)+1}(\underline{\mathscr{K}})$. Thus there is $K_0 \in D^{1+\nu+1}(\underline{\mathscr{K}})$ and $K_0 \neq L_0$. We take K_1, K_2 in $\underline{\mathscr{K}}$ such that $R(K_1, \mathscr{K}^a) \neq R(K_2, \mathscr{K}^a)$ and $K_0 < K_1 < K_2 < L_0$. By Lemma 7.16 there exists an automorphism f of $\mathfrak{A}$ such that $f(K_2) \supseteq \mathrm{conv}(\{a\} \cup K_1)$. Let $K_2' = f(K_2)$; $\mathfrak{K}_2, \mathfrak{K}_2'$ are the submodels of $\mathfrak{A}$ whose universes are K_2 and K_2' respectively.

$P(f^{-1}(a),\mathfrak{A})$ is $K_2,\mathfrak{A}$ isolated, hence by Lemma 7.8(i), $P(f^{-1}(a),\mathfrak{K}_2)$ is isolated in $S_1(T_{\mathfrak{K}_2})$. Hence: (i) $P(a,\mathfrak{K}_2')$ is isolated in $S_1(T_{\mathfrak{K}_2'})$. Let $U = \{K \mid K \in \mathscr{K}^a_{\mathfrak{A}}$ and $a \leqq K \leqq K_1\}$. By Lemma 7.12(ii), $U \in \mathscr{K}_{\kappa,2}$. Hence we conclude (ii): $K_0 \in D^{a_\nu+1+\nu+1}(\mathscr{K}^a_{\mathfrak{K}_2'})$. Let $B' = \bigcup U$, and $C \in B'$, $C = |\mathfrak{K}_2'|_{\psi,a}$. Let $\phi(v_0)$ be a formula which is satisfied by a member of C. Since $\mathfrak{K}_2'$ is ω-saturated and since by Lemma 7.8(i) only countably many types of $S_1(T_{\mathfrak{K}_2'})$ are realized in $\mathfrak{K}_2'$, $S_1(T_{\mathfrak{K}_2'})$ is countable and therefore atomic. Let $\alpha(v_0)$ generate $P(a,\mathfrak{K}_2')$ and let $Q \in S_1(T_{\mathfrak{K}_2'})$ be isolated and $Q \ni \exists x(\alpha(x) \wedge \psi(x,v_0)) \wedge \phi(v_0)$; then it is easily seen that Q is realized in C. From the first part of the proof we now conclude that there is $P \in D^{\nu+1}(S_1(T_{\mathfrak{K}_2'})) \cap P(K_0,\mathfrak{K}_2')$. Let $c \in K_0$ realize P; then by Lemma 7.8(i), $P(c,\mathfrak{A}) \in D^{\nu+1}(S_1(T_{\mathfrak{A}}))$. Since we chose $K_0 \in V$, $P(K_0,\mathfrak{A}) \cap D^{\nu+1}(S_1(T_{\mathfrak{A}})) = \varnothing$, and we arrive at a contradiction. Hence it must be the case that

$$P(L_0,\mathfrak{A}) \cap D^{\nu+1}(S_1(T_{\mathfrak{A}})) \neq \varnothing .$$

We now turn to the case when $R(L_0,V_2) = R(L_0,\mathscr{K}^a)$. Assume again that $D^{\nu+1}(S_1(T_{\mathfrak{A}})) \cup P(L_0,\mathfrak{A}) = \varnothing$, and choose $V \subseteq V_2$, a halfneighbourhood of L_0, such that $P(\bigcup V) \cap D^{\nu}(S_1(T_{\mathfrak{A}}))$ is finite, $P(\bigcup V) \cap D^{\nu+1}(S_1(T_{\mathfrak{A}})) = \varnothing$, and V is closed in $\mathscr{K}^a$. We show the impossibility of the following situation: there exists $K_0 \in D^{a_\nu+1+\nu+1}(\mathscr{K}^a) \cap V, K_1, K_2 \in \mathscr{K}^a, P(K_1,\mathfrak{A}) \subseteq P(K_2,\mathfrak{A}), K_0 < K_1 < K_2$, $R(K_1,\mathscr{K}^a) \neq R(K_2,\mathscr{K}^a)$, and $a \leqq b < K_0$ such that $P(b,\mathfrak{A})$ is conv $(\{a\} \cup K_2),\mathfrak{A}$ isolated. Suppose such a situation does occur. By Lemma 7.16 there is an automorphism f of $\mathfrak{A}$ such that $\operatorname{conv}(\{a\} \cup K_1) \subseteq f(K_2)$. Denote $\mathfrak{K}_2$ and $\mathfrak{K}_2'$ the submodels of $\mathfrak{A}$ whose universes are K_2 and $f(K_2)$ respectively. By Lemma 7.8(i), $P(f^{-1}(b),\mathfrak{K}_2)$ is isolated in $S_1(T_{\mathfrak{K}_2'})$, thus $P(b,\mathfrak{K}_2')$ is also isolated in $S_1(T_{\mathfrak{K}_2'})$. By an argument already used before, we conclude that if $C \in f(K_2)$, C is definable over b in $\mathfrak{K}_2'$ and $\phi(v_0)$ is a formula which is satisfied in $\mathfrak{K}_2'$ by some element of C; then there exists P isolated in $S_1(T_{\mathfrak{K}_2'})$ such that $P \in P(C,\mathfrak{K}_2')$ and $P \ni \phi(v_0)$. Again, by Lemma 7.12(i), (ii), $K_0 \in D^{a_\nu+1+\nu+1}(\mathscr{K}^b_{\mathfrak{K}_2'})$. Now, by the first part of the proof, $P(K_0,\mathfrak{K}_2') \cap D^{\nu+1}(S_1(T_{\mathfrak{K}_2'})) \neq \varnothing$, and by Lemma 7.8 (i) $P(K_0,\mathfrak{A}) \cap D^{\nu+1}(S_1(T_{\mathfrak{A}})) \neq \varnothing$ contradicting the choice of V.

Let $\overline{\mathscr{K}} = D^{a_\nu+1+\nu+1}(\mathscr{K}^a) \cap V$. By Lemma 7.20 there is $\mathscr{K} \subseteq \overline{\mathscr{K}}$ with the following properties:

(i) $L_0 \notin \mathscr{K}$.

(ii) Let $\mathscr{K}_0 = \mathscr{K} \cup \{L_0\}$; then $\mathscr{K}_0$ is complete.

(iii) $\mathscr{K}$ is 2-disjointed.

(iv) $R(L_0,\mathscr{K}_0^{Or}) = R(L_0,\overline{\mathscr{K}})$.

We define $\tau: \mathscr{K}_0 \to \omega_1$:

$$\tau(K) = \min\{R(P, S_1(T_{\mathfrak{A}})) \mid P \in P(\mathrm{conv}(\{a\} \cup K), \mathfrak{A})\}.$$

Obviously, if $K_1 < K_2$ then $\tau(K_1) \geqq \tau(K_2)$. We show that also $\tau(K_1) > \tau(K_2)$. If not, then $\tau(K_1) = \tau(K_2)$. Let $L_1, L_2 \in D^{a_v+1+v+1}(\mathscr{K}^a)$ and $K_1 < L_1 < L_2 < K_2$. There is some $L_3 \in \mathscr{K}^a$ such that $L_1 < L_3 < K_2$ and $R(L_3, \mathscr{K}^a) \neq R(K_2, \mathscr{K}^a)$. Let $b \in \mathrm{conv}(\{a\} \cup K_1)$ and $R(P(b, \mathfrak{A}), S_1(T_{\mathfrak{A}})) = \tau(K_1)$. Thus $R(P(b, \mathfrak{A}), S_1(T_{\mathfrak{A}})) = \tau(K_2)$, hence $P(b, \mathfrak{A})$ is $\mathrm{conv}(\{a\} \cup K_2), \mathfrak{A}$ isolated. But the existence of such b, L_1, L_3, K_2 was proved impossible in the last paragraph, therefore $\tau(K_1) \neq \tau(K_2)$.

Hence τ is order reversing, thus $\mathscr{K}_0^*$ is well ordered. Let $\{K_v\}_{v<\delta+1}$ be an isomorphism between $\mathscr{K}_0^*$ and $\delta+1$. Certainly $L_0 = K_\delta$, thus $\tau(L_0) \geqq \delta$. But $L_0 \in D^{a_v+1}(V)$, hence by Lemma 7.4, $L_0 \in D^{d(v)+1}(D^{a_v+1+v+1}(V))$, therefore $L_0 \in D^{d(v)+1}(\overline{\mathscr{K}})$, so $L_0 \in D^{d(v)+1}(\mathscr{K}_0^{Or})$, so $\delta \in D^{d(v)+1}(\delta+1) \subseteq D^{d(v)+1}(\omega_1)$, but $v+1 \cap D^{d(v)+1}(\omega_1) = \varnothing$ thus $\delta \geqq v+1$ and hence $\tau(L_0) \geqq v+1$. Recalling the definition of τ we deduce that $P(L_0, \mathfrak{A}) \subseteq D^{v+1}(S_1(T_{\mathfrak{A}}))$ in contradiction to our assumption on L_0. We now conclude that $P(L_0, \mathfrak{A}) \cap D^{v+1}(S_1(T_{\mathfrak{A}})) \neq \varnothing$, and the theorem is proved.

We now turn to prove a topological lemma. If X is a topological space and $\mathscr{F}$ is a partition of X we denote the quotient space by $X/\mathscr{F}$. The definition of $X/\mathscr{F}$ can be found in [3, p. 97]. If X is Hausdorff compact and $\mathscr{F}$ consists of closed sets then $X/\mathscr{F}$ is Hausdorff compact.

LEMMA 7.24. *Let X be a countable Hausdorff compact space, $\mathscr{F}$ a partition of X consisting of closed sets; let $M = \bigcup\{R(F) \mid F \in \mathscr{F}\} + 1$, and let $x \in F \in \mathscr{F}$; then $R(x, X) \leqq M \cdot R(F, X/\mathscr{F}) + R(x, F)$.*

PROOF. We prove the lemma by induction on $R(F, X/\mathscr{F})$. Let $R(F, X/\mathscr{F}) = 0$; then F is open in X, $X - F$ is closed in X; by Lemma 7.5, $R(x, X) = R(x, F)$ and the inequality holds. Suppose the inequality holds for every $x \in F \in \mathscr{F}$ such that $R(F, X/\mathscr{F}) < v$. We prove by induction on $R(x, F)$ that if $x \in F$ and $R(F, X/\mathscr{F}) = v$ then again the same inequality holds. Suppose the inequality holds for every $x \in F$ such that $R(F, X/\mathscr{F}) = v$ and $R(x, F) < \xi$, $\xi \geqq 0$, and let $R(x, F) = \xi$ and $R(F, X/\mathscr{F}) = v$. Let U be a neighbourhood of x relative to X such that for every $y \in U \cap F$ if $y \neq x$ then $R(y, F) < \xi$. Let V be a neighbourhood of F relative to $X/\mathscr{F}$ such that for every $G \in V$, $R(G, X/\mathscr{F}) < v$, and let $W = (\bigcup V) \cap U$. W is a neighbourhood of x relative to X. If $y \in W$ and $y \neq x$ either $y \in F$, and then by the induction hypothesis and the choice of U,

$$R(y,X) \leqq M\cdot v + R(y,F) < Mv + \xi$$

or $y \in G \neq F$, and then by the induction hypothesis on $R(G, X/\mathscr{F})$ and the choice of V, $R(y,X) \leqq M\cdot R(G,X/\mathscr{F}) + R(y,G) < M\cdot R(G,X/\mathscr{F}) + M = M\cdot(R(G,X/\mathscr{F}) + 1) \leqq M\cdot v + \xi$. Therefore, for every $y \in W$ if $y \neq x$ then $R(y,X) < M\cdot v + \xi$, thus $R(x,X) \leqq M\cdot v + \xi = M\cdot R(F,X/\mathscr{F}) + R(x,F)$.

Q.E.D

We shall need the following corollaries.

COROLLARY 7.25. (i) *Let X be a countable Hausdorff compact space, $\mathscr{F}$ a partition of X consisting of closed set s; then* $R(X) < (\bigcup_{F\in\mathscr{F}} R(F) + 1)\cdot(R(X/\mathscr{F}) + 1)$.

(ii) *If X, Y are countable Hausdorff compact spaces then* $R(X \times Y) < (R(X) + 1)(R(Y) + 1)$.

PROOF. (i) is proved by taking upper bounds of both sides of the inequality in Lemma 7.24 when x ranges over X.

To prove (ii), let $\mathscr{F} = \{X \times \{y\} \mid y \in Y\}$; then $\mathscr{F}$ is a partition of $X \times Y$ consisting of closed sets, and each $F \in \mathscr{F}$ is homeomorphic to X, hence $R(X \times Y) < (R(X) + 1)\cdot(R(X \times Y/\mathscr{F}) + 1)$. But $(X \times Y)/\mathscr{F}$ is homeomorphic to Y and (ii) is proved.

We shall now partition $S_2(T)$; each element of the partition will again be partitioned. By means of the two partitions and Lemma 7.24 and Corollary 7.25 we shall find an upper bound for $R(S_2(T))$ in terms of $R(S_1(T))$.

Let $\mathfrak{A}$ be an ω-saturated model of T. We first partition $S_2(T)$: if $P \in S_1(T)$ let $S_P = \{Q \mid Q \in S_2(T) \text{ and } Q \supseteq P\}$ and $\mathscr{F} = \{S_P \mid P \in S_1(T)\}$. Certainly $\mathscr{F}$ is a partition of $S_2(T)$ consisting of closed sets. Let $a \in |\mathfrak{A}|$ and $P = P(a, \mathfrak{A})$. We now partition S_P: if $K \in \mathscr{K}^a$ let $S_{P,K} = \{Q \mid Q \in S_P$, and there is some $b \in K$ such that $Q = P(\langle a, b\rangle, \mathfrak{A})\}$.

Let $\mathscr{F}_P = \{S_{P,K} \mid K \in \mathscr{K}^a\}$. By Lemma 7.7(i), $\mathscr{F}_P$ consists of closed subsets of S_P. By Lemma 7.2 it is easy to see that $\mathscr{K}^a$ (with the order topology) is homeomorphic to $S_P/\mathscr{F}_P$. Let $S'_{P,K} = \{Q \mid Q \in S_1(T) \cap P(K, \mathfrak{A})\}$. By Lemmas 7.2 and 7.8(iii) $S'_{P,K}$ (as a subspace of $S_1(T)$) is homeomorphic to $S_{P,K}$.

COROLLARY 7.25 and the above remarks yield:

(1) $R(S_P) < [\bigcup_{K\in\mathscr{K}^a} R(S_{P,K}) + 1]\cdot[R(S_P/\mathscr{F}_P) + 1] \leqq [R(S_1(T)) + 1]\cdot[R(\mathscr{K}^a) + 1]$.

It is again easy to see that the function $P \to S_P$ is a homeomorphism between $S_1(T)$ and $S_2(T)/\mathscr{F}$ hence

$$R(S_2(T)) < \left[\bigcup_{P \in S_1(T)} R(S_P) + 1\right] \cdot [R(S_2(T)/\mathscr{F}) + 1] \tag{2}$$

$$< \left[(R(S_1(T)) + 1) \cdot \left(\bigcup_{a \in A} R(\mathscr{K}^a) + 1\right) + 1\right] \cdot [R(S_1(T)) + 1].$$

From Theorem 7.23 we deduce

$$\text{If } R(S_1(T)) < \nu \text{ then for every } a \in A,\ R(\mathscr{K}^a) < a_\nu. \tag{3}$$

Combining (2) and (3) we obtain Theorem 7.26.

THEOREM 7.26. *If $S_1(T)$ is countable and $R(S_1(T)) < \nu$ then*

$$R(S_2(T)) < [(\nu + 1) \cdot (a_\nu + 1) + 1] \cdot (\nu + 1).$$

It is now easy to find similar upper bounds for $S_n(T)$ when $n > 2$. By Lemma 7.9, $S_n(T)$ is the union of a finite number of closed subsets each of which is homeomorphic to a subset of $(S_2(T))^{n-1}$. By Lemma 7.5, $R(S_n(T))$ equals the maximum rank of the mentioned closed sets. Thus $R(S_n(T)) \leqq R((S_2(T))^{n-1})$; combining the last inequality with Theorem 7.26 and Corollary 7.25 we obtain Theorem 7.27.

THEOREM 7.27. *If $S_1(T)$ is countable, $R(S_1(T)) < \nu$, and $n \geqq 2$ then*

$$R(S_n(T)) < [[(\nu + 1) \cdot (a_\nu + 1) + 1] \cdot (\nu + 1)]^{n-1}.$$

REMARK. It is easy to see that $[(\nu + 1)(a_\nu + 1) + 1] \cdot (\nu + 1) \leqq \nu^4 \cdot 4 + 20$. Thus if $R(S_1(T)) < \nu$ then $R(S_2(T)) < \nu^4 \cdot 4 + 20$.

ACKNOWLEDGEMENT

I am indebted to Professor S. Shelah for corrections and simplifications of several proofs; the present proofs of Theorem 4.4 and Lemma 6.1 are due to him.

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INSTITUTE OF MATHEMATICS
THE HEBREW UNIVERSITY OF JERUSALEM
JERUSALEM, ISRAEL